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Lecture 6. Picard Iterates. Pricing of options in the simple model.

The theorem on existence and uniqueness of solutions for differential equations has an amazing proof, using two concepts we talked about before: recursive sequences and the contraction fixed point theorem. Here's the outline of the proof.

We need to solve the equation $f' = G(t, f)$ subject to the initial data $f(0) = f_0$. The first step is to change the problem into an equivalent one by integration:

$$f(t) = f_0 + \int_0^t G(s, f(s))\, ds.$$

Here we have integrated on both sides, and used the initial data. It is a good idea to run an example parallel to the theory. Take $G(t, z) = z$, $f(0) = 1$. The differential equation in this case is $f' = f$, $f(0) = 1$, and even before we start we know the answer must be $f(t) = e^t$. The integral equation equivalent to the IVP is

$$f(t) = 1 + \int_0^t f(s)\, ds.$$

The key idea now is to interpret the right hand side of the above equation as a transformation, changing a function g into something else.

$$(Tg)(t) = 1 + \int_0^t G(s, g(s))\, ds.$$

In our example, if we take $g(t) = \sin t$, then

$$(Tg)(t) = 1 + \int_0^t \sin s\, ds = 1 + (-\cos t - \cos 0) = -\cos t.$$

What Picard saw was that the solution f we are looking for is a *fixed point* for the transformation, that is, $f = Tf$.

The next thing Picard noticed was that fixed points usually (but not always) are limits of recursive sequences. He then considered the following recursive sequence:

$$f_0(t) = f_0, \quad f_1(t) = (Tf_0)(t), \quad f_2(t) = (Tf_1)(t), \quad f_{n+1}(t) = (Tf_n)(t).$$

In our example, this means

$$f_0(t) = 1, \quad f_1(t) = 1 + \int_0^t ds = 1 + t, \quad f_2(t) = 1 + \int_0^t (1 + s)\, ds = 1 + t + \frac{t^2}{2},$$

$$f_3(t) = 1 + \int_0^t (1 + s + \frac{s^2}{2})\, ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{6}, \quad f_4(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}.$$

Do you see a pattern? The Picard iterates in this case are the truncations of the Taylor series for e^t, and so they converge to e^t as the iteration is continued.

Why should the iterates converge? Let's calculate the difference between two successive iterates.

$$f_{n+1}(t) - f_n(t) = (Tf_n)(t) - (Tf_{n-1})(t) = \int_0^t G(s, f_n(s)) - G(s, f_{n-1}(s))\, ds.$$

Since by hypothesis G has a derivative in the second variable, we see that if f_n is close to f_{n-1}, then the integrand will be small, and so the whole integral will also be small. That implies that f_{n+1} is close to f_n, and by induction we obtain that the sequence f_n is indeed converging.

To make the above argument precise we would have to quantify what do we mean by saying that f_n is "close to" f_{n-1}. We will not do that here. The important idea to grasp here is that certain expressions can be interpreted as being fixed point problems, and that the solutions to fixed-point problems often arise as recursive sequences converging. To prove that the recursive sequence is indeed converging one needs to write down equations, etc.

Financial interlude, Part 1: Put-Call Parity. One of the consequences of the hypothesis of no-arbitrage is that *if two portfolios have the same payoff (in all possible states of the world) , then they must cost the same.* That stands to reason, for suppose a portfolio θ_1 cost p_1, and a portfolio θ_2 cost p_2, and $p_2 > p_1$, but they always pay the same values, no matter what. Then I never want to buy the second portfolio, since I can get the same returns for a lower price with the first portfolio. But more than that, there would be an arbitrage in the market: short (sell) portfolio 2, collect p_2 now, and long (buy) portfolio 1 (which costs p_1). You make $p_2 - p_1 > 0$ now. Since these portfolios make exactly the same amount, whatever you lose in portfolio 2 will be perfectly offset by what you will make in portfolio 1, and vice-versa. Thus you can never lose money in the future, and this would characterize an arbitrage in this market.

One interesting consequence of the above is the **put-call parity** formula. Suppose we have a stock whose value $S = S(t)$ changes with time t. We also have, for that stock, calls and puts, both with strike price E and expiry date T. The call pays (and therefore, costs) $C = C(t)$, the put $P = P(t)$. We adopt the following portfolio: Buy the stock, buy a put, and sell the call. The value (for us) of this portfolio is $S + P - C$, and this value changes with time, depending on how the stock performs.

What is our payoff at expiry ($T = t$)?

The payoff may depend on the value of the stock. If the stock tanked and $S \leq E$, then the stock pays S, the call is worthless (so $C = 0$), and the put pays $E - S$, therefore our portfolio is paying the quantity E. On the other hand, if $S > E$, then the stock pays S, the put is worthless ($P = 0$), and the call will set us back the quantity $S - E$. Our portfolio is worth $S - (S - E) = E$. We conclude that this portfolio must, in all events, pay off the value E to us at expiry.

But now consider a portfolio consisting only of bonds, and also paying E at expiry. Since these portfolios pay the same, they must cost the same. Since the bond must cost $Ee^{-r(T-t)}$, we conclude that

$$S + C - P = Ee^{-r(T-t)}.$$

This is the Put-Call Parity formula.

Financial interlude, Part 2. In our basic example, let's now add a new security V to our market. If the new security V is independent of the security S we already have in the market, then it is very likely that the number of future states would have to increase to reflect this independence. (Think of the earthquake and coin flip examples.) On the other hand, if the new security V derives its value from that of security S, then there will be no new future states. In that case we say that V is a financial derivative of S, and S is the *underlying asset*.

The price of V will be denoted by V_0, and the future states of V will have payoffs V_1 and V_2. Since V is a derivative of S, the values V_1 and V_2 depend directly on the values of S. What that means is that there's a function $\Lambda(S) = V$, called the *payout function*, which gives V its value according to that of S.

Example. Let S be a stock with current value $S_0 = 100$, and future states valued at $S_1 = 90$, $S_2 = 110$. Let V be a call option with strike price $E = 100$. We already saw that the value of a call option at the expiry date depends on the value of the underlying asset, given by the formula $\Lambda(S) = \max\{0, S - E\}$. Therefore $V_1 = \max\{0, 90 - 100\} = 0$, and $V_2 = \max\{0, 110 - 100\} = 10$.

Since we did not increase the states of the world, the prices in this market are completely determined by the payout matrix when applied to the state-price vector ψ, which we already calculated last class. In symbols,

$$\begin{pmatrix} e^{-rT} \\ S_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ S_1 & S_2 \\ V_1 & V_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

But this gives us the price of the derivative: $V_0 = \psi_1 V_1 + \psi_2 V_2$. Substituting ψ_1 and ψ_2 we obtain

$$V_0 = \frac{e^{-rT} S_2 - S_0}{S_2 - S_1} V_1 + \frac{S_0 - S_1 e^{-rT}}{S_2 - S_1} V_2.$$

Back to math. The differential equations we have seen so far are said to be of *order 1*, because the first derivative is the highest derivative showing up in the equation. We can also study equations of *order 2*, like

$$f''(t) + f(t) = 0 \qquad \text{or} \qquad t^2 f''(t) + t\, f'(t) + f(t) = 0.$$

These equations are *much* harder to solve (in general) than equations of order 1, but there are all sorts of special cases that actually happen in practice.

Here are some things to keep in mind for linear equations of order 2:

- The equation will have two solutions $f_1(t)$ and $f_2(t)$ that are not a constant multiple of each other (we call them *linearly independent*), and every other solution is of the form $Af_1(t) + Bf_2(t)$.

- In general it is very hard to figure out even *one* solution of the differential equation, but if by any chance you *do* know one solution, then it is easy to obtain another one.

- The IVP needs two pieces of data: $f(t_0)$ and $f'(t_0)$.

Problems.

1. Solve the differential equation $f'(t) = t\,f(t)$ with initial condition $f(0) = 1$.

2. Obtain the first four Picard iterates for the previous problem. Here $G(t, z) = tz$, and I want you to find $f_0(t)$, $f_1(t)$, $f_2(t)$, and $f_4(t)$.

3. Obtain the fourth Taylor polynomial for the solution in problem 1, and compare with your answer in problem 2.

4. Consider the second order equation

$$t^2\,f''(t) + t\,f'(t) + f(t) = 0.$$

Show that the function $f_1(t) = \cos(\ln t)$ is a solution of this equation, then guess what another solution $f_2(t)$ would be, and show that your guess is correct. (Note: your guess can't be a multiple of $f_1(t)$!)

5. Take $S_0 = 100$, $S_1 = 90$, $S_2 = 110$, $E = 100$, $r = 6\%$ per annum, $T =$ three months. Let V be a call option for the underlying asset S. Then $V_1 = 0$ and $V_2 = 10$. Find V_0. You may assume $T = 3/12$ of a year.

6. In problem 5, if the expiry date increases, should V_0 increase, decrease, or stay the same? (Use the formula to answer this question.) Can this answer be justified *without* using the formula? (Purely in economics terms.)

7. In problem 5, if you increase the strike price, what happens to V_0? Can you justify it without using the formula?