Prove that there exists a surjection from $\mathcal{P}(\mathbb{N})$ onto ω_1.

The solution may have some aspects in common with the proof of Hartog's theroem.

[we may not use the Axiom of Choice]

Notation and Definitions

Notation: ω_1 denotes the first uncountable ordinal.

$\leq$ is a well ordering of ω_1 with the property that $\mathrm{seg}_{\omega_1}(\alpha)$ is countable for all $\alpha \in \omega_1$

Here is Hartog's Theorem:

[from Notes on Set Theory -Yiannis Moschovakis]
[Let me know if you have a question on notation; I have a pdf file of this book]

7.34. Hartogs' Theorem. *There is a definite operation $\chi(A)$ which associates with each set A, a well ordered set*

$$\chi(A) = (h(A), \leq_{\chi(A)}),$$

such that $h(A) \not\leq_c A$, i.e., there exists no injection $\pi : h(A) \rightarrowtail A$. Moreover, $\chi(A)$ is $\leq_o$-minimal with this property, i.e., for every well ordered set W,

$$\textit{if } W \not\leq_c A, \textit{ then } \chi(A) \leq_o W. \tag{7-27}$$

Proof. First set

$$\mathrm{WO}(A) =_{\mathrm{df}} \{U \mid U = (\mathrm{Field}(U), \leq_U) \text{ is a well ordered set with } \mathrm{Field}(U) \subseteq A\}, \tag{7-28}$$

and let $\sim_A$ be the restriction of the definite condition $=_o$ to $\mathrm{WO}(A)$,

$$U \sim_A V \iff_{\mathrm{df}} U, V \in \mathrm{WO}(A) \;\&\; U =_o V.$$

Clearly $\sim_A$ is an equivalence relation on $\mathrm{WO}(A)$, and we set

$$h(A) =_{\mathrm{df}} [\![\mathrm{WO}(A)/{\sim_A}]\!] \subseteq \mathcal{P}(\mathrm{WO}(A)). \tag{7-29}$$

We order the equivalence classes in $h(A)$ by their "representatives",

$$[U/{\sim_A}] \leq_{\chi(A)} [V/{\sim_A}] \iff_{\mathrm{df}} U \leq_o V; \tag{7-30}$$

this makes sense because if

$$[U/{\sim_A}] = [U'/{\sim_A}],\ [V/{\sim_A}] = [V'/{\sim_A}], \text{ and } U \leq_o V,$$

then $U' =_o U \leq_o V =_o V'$. The fact that $\leq_{\chi(A)}$ is a wellordering of $h(A)$ follows easily from the general properties of $\leq_o$, **7.31** and **7.33**. Taking the negation of both sides of (7-30) we infer its strict version,

$$V <_o U \iff [V/\sim_A] <_{\chi(A)} [U/\sim_A] \quad (U, V \in \mathrm{WO}(A)). \tag{7-31}$$