1. Operator Algebra. Evaluate the following expressions:

a. (10 pts) $e^{-ia\hat{p}_x/\hbar}\,\hat{x}\,e^{ia\hat{p}_x/\hbar}$. (The final result should not depend on $\hat{p}_x$.) What is the physical interpretation of your results?

b. (10 pts) $e^{-i\mathbf{a}\cdot\hat{\mathbf{p}}/\hbar}\,\hat{\mathbf{L}}\,e^{i\mathbf{a}\cdot\hat{\mathbf{p}}/\hbar}$, where $\hat{\mathbf{L}}$ is the angular momentum operator. What is the physical interpretation of your results?

2. Neutrino Oscillation made oversimple. Neutrinos come in three varieties that we know of: the electron neutrino $|\nu_e\rangle$, the tau neutrino $|\nu_\tau\rangle$, and the muon neutrino (which is irrelevant to this problem.) Nuclear fusion in the sun's interior produces electron neutrino. A major mystery of modern physics is that we observe too few of these neutrinos on earth. This problem describes (in a super simplified way) one possible explanation.

Suppose the Hamiltonian of the sun (which describes the interaction of neutrinos with the rest of the sun) can be described as follows:

$$\hat{H}|\nu_e\rangle = a|\nu_e\rangle + b|\nu_\tau\rangle \tag{1}$$

$$\hat{H}|\nu_\tau\rangle = b|\nu_e\rangle + a|\nu_\tau\rangle. \tag{2}$$

a. (10 pts) Find the energy eigenstates of this Hamiltonian in terms of the "basis vectors" $|\nu_e\rangle$ and $|\nu_\tau\rangle$, and the corresponding energy eigenvalues.

b. (5 pts) Suppose that $t = 0$, an electron neutrino $|\nu_e\rangle$ is produced in the center of the sun. What is the state at time $t > 0$, in terms of the parameters a and b?

c. (5 pts) What is the probability that the original electron neutrino will be observed as a tau neutrino at time t?