UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS & STATISTICS

> SEMESTER 1
> 2002/2003

MAS332

Number Theory

Time allowed: 1 Hour 30 minutes

Credit will be given for ALL answers to questions in Section A and the best TWO answers to questions in Section B.

No credit will be given for other answers, and students are strongly advised not to spend time producing answers for which they will receive no credit.

There are FOUR questions in Section A and THREE questions in Section B. Marks allocated to each question are indicated. However, you are advised that marks indicate the relative weight of individual questions; they do not correspond directly to marks on the University scale.

SECTION A

A1. Prove that there are infinitely many primes of the form $4k - 1$.

[10 marks]

A2. Denote by $\sigma(n)$ the sum of all divisors of n.

Prove that $\sigma(n) = \prod\limits_{i=1}^{r}\left(\dfrac{p_i^{\alpha_i+1} - 1}{p_i - 1} \right)$

where $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ is the collected prime factorization of n.

[10 marks]

A3. Denote by $\phi(n)$ the number of integers in the range 1 to n which are coprime to n. State a formula for $\phi(n)$ in terms of collected prime factorization. <u>Hence</u> find the all positive numbers n with

$$\phi(5n) = 2\phi(3n).$$

[10 marks]

A4. State the law of quadratic reciprocity and find $\left(\dfrac{557}{2003} \right)$. Is there a solution to the equation $x^2 \equiv 557 \pmod{2003}$?

[10 marks]

SECTION B

B5.

(a)   (i)   State and <u>prove</u> the Chinese Remainder Theorem.

      (ii)   Find the 2 smallest positive integer solutions of the simultaneous set of congruence equations:

$$2x \equiv 3 \pmod{5}$$

$$3x \equiv 4 \pmod{7}$$

$$x \equiv 5 \pmod{8}$$

(b)   Let p be a prime and α a positive integer. How many solutions are there to the equation $x^2 - x \equiv 0 \pmod{p^{\alpha}}$?

(c)   Let n and m be coprime integers. Show $x^2 - x \equiv 0 \pmod{nm}$ if and only if $x^2 - x \equiv 0 \pmod{n}$ and $x^2 - x \equiv 0 \pmod{m}$.

(d)   How many solutions are there to the equation $x^2 - x \equiv 0 \pmod{N}$ where N has collected prime factorization $N = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$.

[30 marks]

B6.

(a) State a formula for $\tau(n)$, the number of divisions of n, in terms of the collected prime factorization of n.

(b) Define the term _multiplicative function_.

(c) Suppose that f and g are multiplicative functions. Prove that the function h defined by

$$h(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

is also multiplicative.

(d) Find a formula for $q(n)$ in terms of the collected prime factorization for n, where

$$q(n) = \sum_{d|n} \tau(d).$$

(e) Find the smallest positive solution to $q(n) = 45$.

[30 marks]

B7.

(a) Define the Riemann Zeta function $\zeta(s)$ as an infinite sum. Prove that

$$\zeta(s) = \prod_{p_i \text{ prime}} \left(1 - p_i^{-s}\right)^{-1} \qquad (\text{for } s > 1).$$

(b) Define the Möbius function $\mu(n)$. Prove that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad (\text{for } s > 1).$$

(c) State Dirichlet's Muliplication Theorem for Dirichlet Series.

(d) Use the Dirichlet Muliplication Theorem to find the sum of the following Dirichlet Series in terms of $\zeta(s)$ and $\zeta(s-1)$:

$$S(s) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} \qquad (\text{for } s > 2).$$

(e) Write $S(s)$ as an infinite product over primes.

[30 marks]