

Prove that the subtraction operation defined in Proposition 2.7.9 coincides with the following operation defined by recursion:

$$\begin{aligned}\alpha \dot{-} \alpha &= 0 \\ \beta^+ \dot{-} \alpha &= (\beta \dot{-} \alpha)^+ \text{ for } \alpha \leq \beta \\ \gamma \dot{-} \alpha &= \sup\{\beta \dot{-} \alpha : \beta < \gamma\} \text{ when } \gamma \text{ is a limit.}\end{aligned}$$

Proposition 2.7.9. For any pair of ordinals α and β such that $\alpha \leq \beta$, there is a unique ordinal γ such that $\alpha + \gamma = \beta$. So we can write (unambiguously) $\gamma = \beta - \alpha$, when $\beta \geq \alpha$.

Proof. For existence, observe that if $\alpha = \text{ord}(A)$ and $\beta = \text{ord}(B)$ then $A \triangleleft B$. Let $\bar{A} \subseteq_e B$ be the initial segment corresponding to A . Then $C = B \setminus \bar{A}$ is a well-ordered set and $B = \bar{A} \cup C$ so $\text{ord}(\bar{A}) + \text{ord}(C) = \text{ord}(B)$. But $\text{ord}(\bar{A}) = \alpha$ and so we can set $\gamma = \text{ord}(C)$.

For uniqueness, suppose γ' is another solution and suppose wlog that $\gamma < \gamma'$. Then there is a $\delta > 0$ such that $\gamma + \delta = \gamma'$. But then $\beta = \alpha + \gamma' = \alpha + \gamma + \delta > \alpha + \gamma = \beta$, a contradiction. \square
So this says that to compute $\beta - \alpha$, we remove α from the beginning of β .