

* Exercise 5.2.2. * (a) Show that for any normalized $|\psi\rangle$, $\langle\psi|H|\psi\rangle \geq E_0$, where E_0 is the lowest-energy eigenvalue. (Hint: Expand $|\psi\rangle$ in the eigenbasis of H .)

(b) Prove the following theorem: Every attractive potential in one dimension has at least one bound state. Hint: Since V is attractive, if we define $V(\infty)=0$, it follows that $V(x) = -|V(x)|$ for all x . To show that there exists a bound state with $E < 0$, consider

$$\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$$

and calculate

$$E(\alpha) = \langle\psi_\alpha|H|\psi_\alpha\rangle, \quad H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - |V(x)|$$

Show that $E(\alpha)$ can be made negative by a suitable choice of α . The desired result follows from the application of the theorem proved above.

* Exercise 5.2.3. * Consider $V(x) = -aV_0\delta(x)$. Show that it admits a bound state of energy $E = -ma^2V_0^2/2\hbar^2$. Are there any other bound states? Hint: Solve Schrödinger's equation outside the potential for $E < 0$, and keep only the solution that has the right behavior at infinity and is continuous at $x=0$. Draw the wave function and see how there is a cusp, or a discontinuous change of slope at $x=0$. Calculate the change in slope and equate it to

$$\int_{-\epsilon}^{+\epsilon} \left(\frac{d^2\psi}{dx^2}\right) dx$$

(where ϵ is infinitesimal) determined from Schrödinger's equation.