The translation operator $S(\vec{a})$, for arbitrary constant real vector $\vec{a}$, is defined such that

$$\text{if} \quad \langle \vec{x} | \phi \rangle = \phi(\vec{x}), \quad \text{then} \quad \langle \vec{x} | \, S(\vec{a}) \, | \phi \rangle = \phi(\vec{x} - \vec{a}),$$

$$\text{so that} \quad \langle \psi | \, S \, | \phi \rangle = \int \psi^*(\vec{x}) \phi(\vec{x} - \vec{a}) d\vec{x}.$$

a) Show, using the Taylor expansion of $\phi(\vec{x} - \vec{a})$ about $\vec{x}$, that $S(\vec{a})$ may be expressed in terms of the momentum operator $\vec{p} = -i\hbar\vec{\nabla}$. Sum the expansion to give a simple closed-form expression for $S(\vec{a})$.

b) Show, from the definition of the Hermitian conjugate of S, $\langle \psi | \, S \, | \phi \rangle^* = \langle \phi | \, S^\dagger \, | \psi \rangle$, for all kets $|\phi\rangle$ and $|\psi\rangle$, that S is unitary ($S^\dagger = S^{-1}$). Give a simple closed-form expression for $S^{-1}(a)$.

c) Evaluate $[x_i, S(\vec{a})]$, $i = 1, 3$.

d) Using (c), show how the expectation value $\langle \vec{x} \rangle$ changes under translation.