Exercise 5.1.3 (*Another Way to Do the Gaussian Problem*). We have seen that there exists another formula for $U(t)$, namely, $U(t)=e^{-iHt/\hbar}$. For a free particle this becomes

$$U(t)=\exp\left[\frac{i}{\hbar}\left(\frac{\hbar^2 t}{2m}\frac{d^2}{dx^2}\right)\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\hbar t}{2m}\right)^n\frac{d^{2n}}{dx^{2n}} \tag{5.1.18}$$

Consider the initial state in Eq. (5.1.14) with $p_0=0$, and set $\Delta=1$, $t'=0$:

$$\psi(x,0)=\frac{e^{-x^2/2}}{(\pi)^{1/4}}$$

nd $\psi(x,t)$ using Eq. (5.1.18) above and compare with Eq. (5.1.15).
Hints: (1) Write $\psi(x,0)$ as a power series:

$$\psi(x,0)=(\pi)^{-1/4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(2)^n}$$

(2) Find the action of a few terms

$$1,\qquad \left(\frac{i\hbar t}{2m}\right)\frac{d^2}{dx^2},\qquad \frac{1}{2!}\left(\frac{i\hbar t}{2m}\frac{d^2}{dx^2}\right)^2$$

etc., on this power series.
(3) Collect terms with the same power of x.
(4) Look for the following series expansion in the coefficient of x^{2n}:

$$\left(1+\frac{it\hbar}{m}\right)^{-n-1/2}=1-(n+1/2)\left(\frac{i\hbar t}{m}\right)+\frac{(n+1/2)(n+3/2)}{2!}\left(\frac{it\hbar}{m}\right)^2+\cdots$$

(5) Juggle around till you get the answer.

Exercise 5.1.4: A Famous Counterexample. Consider the wave function

$$\psi(x,0)=\sin\left(\frac{\pi x}{L}\right),\qquad |x|\le L/2$$
$$=0,\qquad |x|>L/2$$

It is clear that when this function is differentiated any number of times we get another function confined to the interval $|x|\le L/2$. Consequently the action of

$$U(t)=\exp\left[\frac{i}{\hbar}\left(\frac{\hbar^2 t}{2m}\right)\frac{d^2}{dx^2}\right]$$

on this function is to give a function confined to $|x|\le L/2$. What about the spreading of the wave packet?