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5. Find the Fourier series of the periodic extension of the function e^x defined on]-a,a[. Use this result to find the Fourier series of the periodic extension of the function cosh x defined on]-a,a[.

Ans. $1\frac{1}{2}$ $e^x = a_0 + \sum_{n\geq 1} a_n \cos\frac{n\pi x}{a} + b_n \sin\frac{n\pi x}{a}$ (1) $a_0 = \frac{1}{a}\sinh a$

$a_n = \frac{2a\cos n\pi \sinh a}{(a^2+n^2\pi^2)}$ $1\frac{1}{2}$ $b_n = \frac{-2n\pi\cos n\pi \sinh a}{(a^2+n^2\pi^2)}$ $\cosh(x) = \frac{e^x+e^{-x}}{2} = A_0 + \sum_{n\geq 1} A_n \cos\frac{n\pi x}{a} + B_n \sin\frac{n\pi x}{a}$

(1) $\boxed{A_0 = a_0 \quad A_n = a_n \quad B_n = 0}$

Section C

6. Classify the following equations

$2\frac{1}{2}$ (a) $u_{xx} + (x^2 - x)^{1/2}\, u_{xy} + u_{yy} = 0$

$2\frac{1}{2}$ (b) $u_{xx} + \sin y\; u_{yy} = 0$

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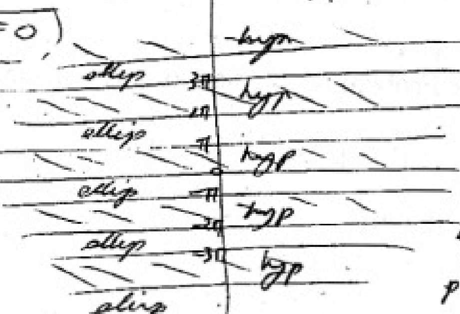
Ans.

(a) hyp. $x^2-x>4$
ellip. $x^2-x<4$
par. $x^2-x=4$

(b)



$\sin y < 0$ hyp
$\sin y = 0$ p.
$\sin y > 0$ ellip

hyp. $]2n\pi, (2n+1)\pi[$
ellip. $](2n-1)\pi, 2n\pi[$
para $\{n\pi\}$

7.Solve the heat equation $u_{xx} = k^{-1} u_t \quad 0 < x < a, t > 0$ with the conditions
$u(0,t) = 0 \quad u_x(a,t) = 0 \quad 0<t \qquad u(x,0) = T_0 \quad 0<x<a$
where T_0 is a constant.

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Ans.

$$u(x,t) = \frac{4T_0}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)}\sin\left(\frac{(2m-1)\pi x}{2a}\right)\exp\left\{-\left[\frac{(2m-1)\pi}{2a}\right]^2 kt\right\}$$

8. Solve the following wave problem $u_{xx} = c^{-2} u_{tt}$, $0<x<a \quad t>0$
witth $u(0,t) = 0 \quad u(a,t) = 0 \quad t>0$ and

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$$u(x,0) = \begin{cases} 2x & 0<x<a/2 \\ 2(a-x) & a/2<x<a \end{cases} \qquad u_t(x,0) = 0 \qquad 0<x<a$$

Ans.

$$u(x,t) = \frac{a^2}{\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{cn\pi t}{a}\right)$$

9. Solve the rectangular drum problem $\quad u_{xx} + u_{yy} = c^{-2} u_{tt} \quad 0<x<a \quad 0<y<b \quad t>0$
$u(0,y,t) = 0 \qquad u(a,y,t) = 0 \quad 0<y<b \quad t>0 \qquad u(x,0,t) = 0 \quad u(x,b,t) = 0 \quad 0<x<a \quad t>0$

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$u(x,y,0) = f(x,y) \qquad u_t(x,y,0) = 0.$ Find the solution explicitly in the case
$u(x,y,0) = xy$ and $u_t(x,y,0) = 0.$ $\quad u(x,y) = \sum_{m,n\geq 1}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\left[a_{mn}\cos(\lambda_{mn}ct) + b_{mn}\sin(\lambda_{mn}ct)\right]$

Ans. $a_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy \qquad \lambda^2_{mn} = \nu_m^2 + \mu_n^2 = \frac{\pi^2}{a^2b^2}(\ldots)$

example $\boxed{u(x,y) = \frac{4ab}{\pi^2}\sum_{m,n\geq 1}\frac{(-1)^{m+n}}{nm}\cos(\lambda_{mn}ct)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}}$

10. Solve the potential equation on the annulus
$r\partial_r(ru_r) + u_{\theta\theta} = 0 \quad a<r<b$ with $u(a,\theta) = 0 \quad u(b,\theta) = \theta$
What additional assumptions are necessary to get a unique solution to this problem which is smooth and bounded on the annulus?

Ans. $u(r,\pi) = u(r,-\pi) \qquad u_\theta(r,\pi) = u_\theta(r,-\pi)$

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$$u(r,\theta) = 2\sum\frac{b^n(-1)^{n+1}}{n(b^{2n}-a^{2n})}\left(r^n - a^{2n}r^{-n}\right)\sin n\theta$$