**ATTEMPT ALL QUESTIONS, BUT AT LEAST ONE FROM EACH SECTION. PLEASE ATTACH YOUR SCRATCH.**

## Section A

1. Use Frobenius's method to find a solution about zero to

$r^2 R'' + r R' - \mu^2 R + r^2 R = 0$ $\quad R = \sum c_i r^{\alpha + i}$ regular sing at $r = 0$ (1)

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where $R = R(r)$.

**Ans.** $\boxed{\alpha = \pm \mu}$ (2) $R = \sum_{n \geq 0} a_{2n} r^{2m+\mu}$ $\quad a_{2m} = \dfrac{(-1)^m}{2^{2m} m! (\mu+m)(\mu+m-1)\cdots(\mu+1)}$ (2)

2. Find a solution about zero to the equation

$(1 - x^2)y'' - 2xy' + \mu^2 y = 0 \quad -1<x<1$

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(a) Is it necessary to use Frobenius's method for this problem? (b) What values of $\mu$ are necessary to ensure that the solutions obtained do not diverge at $x = \pm 1$? (c) Write down the solutions in this case.

**Ans.** Series soln. $y = \sum a_i x^i$ (1) $\quad a_{n+2} = \dfrac{-a_n(\mu^2 - n(n+1))}{(n+2)(n+1)}$ (1/2)

$\lim_{n \to \infty} \left| \dfrac{\mu^2 - n(n+1)}{(n+2)(n+1)} \right| = 1$ diverges $|x|>1$

but if $\mu^2 = k(k+1)$ then series terminates (2 1/2)

## Section B

3. Find the Fourier integral representation of the following function

$$f(x) = \begin{cases} \sin x & -\pi < x < \pi \\ 0 & |x| > \pi \end{cases}$$

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**Ans.** $f(x) = \int_0^\infty B(\lambda) \sin \lambda x \, d\lambda + \int_0^\infty A(\lambda) \cos \lambda x \, d\lambda$ (1) $\quad B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \lambda x \, dx$

$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \lambda x \, dx = 0$ (1) $\quad B(\lambda) = \begin{cases} 1 & \lambda = 1 \\ \frac{2 \sin \lambda \pi}{\pi(1-\lambda^2)} & \lambda \neq 1 \end{cases}$ ← both consist (2) $\quad \boxed{f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda \pi}{(1-\lambda^2)} \sin \lambda x \, d\lambda}$

4. Find the Fourier cosine series of the function $|\sin x|$ in the interval $]-\pi, \pi[$. Use it to find the sums

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$$\Sigma^\infty_{n=1} \; 1/(4n^2 - 1) \qquad \text{and} \qquad \Sigma^\infty_{n=1} \; (-1)^n/(4n^2 - 1)$$

**Ans.**

$\sin x = a_0 + \sum_{n \geq 1} a_n \cos n x \qquad a_0 = \frac{1}{\pi} \int_0^\pi \sin x \, dx = \frac{2}{\pi}$

$a_n = \frac{1}{\pi} \int_0^\pi (\sin(n+1)x + \sin(1-n)x) \, dx \underset{\text{even}}{\overset{\text{odd} = 0}{=}} \frac{-1}{\pi} \left[ \frac{1}{(2m+1)}(-2) + \frac{1}{(1-2m)}(-2) \right]$

$a_{2m} = \dfrac{-4}{\pi(4m^2-1)}$ $\quad \boxed{\sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m \geq 1} \frac{1}{(4m^2-1)} \cos 2mx}$ (3)

$\boxed{x = 0 \quad \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} = \frac{1}{2}}$ (1) $\qquad \boxed{x = \frac{\pi}{2} \quad \sum \frac{(-1)^n}{(4n^2-1)} = \frac{1}{2} - \frac{\pi}{4}}$ (1)