Let A be a real matrix having the upper triangular block structure

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ 0 & A_{22} & A_{23} & \cdots & A_{2n} \\ 0 & 0 & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{nn} \end{bmatrix} \tag{1}$$

in which each A_{ij} is a 2×2 matrix, i.e.:

$$A = \begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{11} & \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{12} & \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{13} & \cdots & \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}_{1n} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{21} & \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}_{22} & \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}_{23} & \cdots & \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}_{2n} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{31} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{32} & \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}_{33} & \cdots & \begin{bmatrix} i_{11} & i_{12} \\ i_{21} & i_{22} \end{bmatrix}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{n1} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{n2} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{n3} & \cdots & \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}_{nn} \end{bmatrix} \tag{2}$$

Note that $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{11} = A_{11}$, $\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}_{22} = A_{22}$, etc.

Thus we have that

$$A = \begin{bmatrix} a_{11_{11}} & a_{12_{11}} & b_{11_{12}} & b_{12_{12}} & c_{11_{13}} & c_{12_{13}} & \cdots & \cdots & d_{11_{1n}} & d_{12_{1n}} \\ a_{21_{11}} & a_{22_{11}} & b_{21_{12}} & b_{22_{12}} & c_{21_{13}} & c_{22_{13}} & \cdots & \cdots & d_{21_{1n}} & d_{22_{1n}} \\ 0_{11_{21}} & 0_{12_{21}} & e_{11_{22}} & e_{12_{22}} & f_{11_{23}} & f_{12_{23}} & \cdots & \cdots & g_{11_{2n}} & g_{12_{2n}} \\ 0_{21_{21}} & 0_{22_{21}} & e_{21_{22}} & e_{22_{22}} & f_{21_{23}} & f_{22_{23}} & \cdots & \cdots & g_{21_{2n}} & g_{22_{2n}} \\ 0_{11_{31}} & 0_{12_{31}} & 0_{11_{32}} & 0_{12_{32}} & h_{11_{33}} & h_{12_{33}} & \cdots & \cdots & i_{11_{3n}} & i_{12_{3n}} \\ 0_{21_{31}} & 0_{22_{31}} & 0_{21_{32}} & 0_{22_{32}} & h_{21_{33}} & h_{22_{33}} & \cdots & \cdots & i_{21_{3n}} & i_{22_{3n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0_{11_{n1}} & 0_{12_{n1}} & 0_{11_{n2}} & 0_{12_{n2}} & 0_{11_{n3}} & 0_{12_{n3}} & \cdots & \cdots & j_{11_{nn}} & j_{12_{nn}} \\ 0_{21_{n1}} & 0_{22_{n1}} & 0_{21_{n2}} & 0_{22_{n2}} & 0_{21_{n3}} & 0_{22_{n3}} & \cdots & \cdots & j_{21_{nn}} & j_{22_{nn}} \end{bmatrix} \quad (3)$$

Note that (1), (2) and (3) above are equivalent.

(a) Prove that

$$det(A) = det(A_{11})det(A_{22})\cdots det(A_{nn}) = det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{11}\right) det\left(\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}_{22}\right) \cdots det\left(\begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}_{nn}\right)$$

Proof:

First we note that

$$det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{11}\right) = (a_{11}\cdot a_{22} - a_{21}\cdot a_{12}), \qquad det\left(\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}_{22}\right) = (e_{11}\cdot e_{22} - e_{21}\cdot e_{12})$$

$$det\left(\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}_{33}\right) = (h_{11}\cdot h_{22} - h_{21}\cdot h_{12}),\ \ldots,\ det\left(\begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}_{nn}\right) = (j_{11}\cdot j_{22} - j_{21}\cdot j_{12})$$

We also note that, in general for $A^{n\times n}$, $det\left(A^{n\times n}\right) = a_{11}\alpha_{11} + a_{12}\alpha_{12} + \ldots + a_{1n}\alpha_{1n}$. We start with a_{11} and we need to find α_{11}. α_{11} is a cofactor and as such, $\alpha_{11} = (-1^2)\beta_{11}$, where β_{11} is the determinant of the $(n-1)\times(n-1)$ matrix created by deleting the first row and first column of A. Then from (3) we have

$$a_{11} \cdot det\left(\begin{bmatrix} a_{22_{11}} & b_{21_{12}} & b_{22_{12}} & c_{21_{13}} & c_{22_{13}} & \cdots & \cdots & d_{21_{1n}} & d_{22_{1n}} \\ 0_{12_{21}} & e_{11_{22}} & e_{12_{22}} & f_{11_{23}} & f_{12_{23}} & \cdots & \cdots & g_{11_{2n}} & g_{12_{2n}} \\ 0_{22_{21}} & e_{21_{22}} & e_{22_{22}} & f_{21_{23}} & f_{22_{23}} & \cdots & \cdots & g_{21_{2n}} & g_{22_{2n}} \\ 0_{12_{31}} & 0_{11_{32}} & 0_{12_{32}} & h_{11_{33}} & h_{12_{33}} & \cdots & \cdots & i_{11_{3n}} & i_{12_{3n}} \\ 0_{22_{31}} & 0_{21_{32}} & 0_{22_{32}} & h_{21_{33}} & h_{22_{33}} & \cdots & \cdots & i_{21_{3n}} & i_{22_{3n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0_{12_{n1}} & 0_{11_{n2}} & 0_{12_{n2}} & 0_{11_{n3}} & 0_{12_{n3}} & \cdots & \cdots & j_{11_{nn}} & j_{12_{nn}} \\ 0_{22_{n1}} & 0_{21_{n2}} & 0_{22_{n2}} & 0_{21_{n3}} & 0_{22_{n3}} & \cdots & \cdots & j_{21_{nn}} & j_{22_{nn}} \end{bmatrix}\right)$$

This is where I need help. I need to know how to write this expansion up rigorously, but in a reasonable amount of time and space.

Through these expansions we can see that

$$det(A) = (a_{11} \cdot a_{22} - a_{21} \cdot a_{12})(e_{11} \cdot e_{22} - e_{21} \cdot e_{12})(h_{11} \cdot h_{22} - h21 \cdot h_{12}) \cdots (j_{11} \cdot j_{22} - j_{21} \cdot j_{12})$$

Which is $det(A_{11})det(A_{22})det(A_{33}) \cdots det(A_{nn})$.

$\blacksquare$

(b) Give a simple procedure for computing the eigenvalues of A including proof.