4. Given the following problem (a variant of the *heat* equation):

$$x^2 \frac{\partial u}{\partial t} = x \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) - u \quad , \quad 0 \le x \le 5 \quad , \quad 0 \le t$$

$$u(0,t) \ \& \ u_x(0,t) \text{ finite} \quad , \quad u(5,t) = 0$$

$$u(x,0) = f(x) = x$$

(a) Separate Variables (with the usual separation constant $-\lambda$) to get 2 ODE's, for $X(x)$ and $T(t)$. (Check your work! This step is critical.)

(b) Show that the eigenvalue problem for $X(x)$ is a Sturm-Liouville problem (what type?); identify the functions $p(x)$, $w(x)$, and $q(x)$.

(c) Use equation (3.10.54) on page 122 (also called the "Rayleigh quotient") to show there are no *negative* eigenvalues. Is *zero* an eigenvalue? [Be sure to use the specific $p(x)$, $w(x)$, $q(x)$ and Boundary Conditions for this problem.]

(d) Find the eigenvalues λ_n and eigenfunctions $X_n(x)$. [Hint: they can be expressed in terms of Bessel functions.] (If you get stuck on this step, just call the eigenfunctions $X_n(x)$ and move on.)

(e) Finish solving the given problem, including finding all coefficients in terms of integrals.

(f) Describe the general behavior of the solution over time: what happens?

(g) [extra credit] Evaluate the integrals.