1. A plane wave $A\exp\{\mathrm{i}(\mathbf{k}_0\cdot\mathbf{r}-\omega t)\}$ is incident upon a regular lattice having identical atoms situated at points $\mathbf{r}_n$ relative to the origin O. Consider the interaction of the wave with a particular atom leading to an elastically scattered wave which reaches a detector D, distant $r_\mathrm{D} \gg 2\pi/k_0$ from O. Show that the total scattered amplitude S reaching D from all atoms may be written as

$$S = fA\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_\mathrm{D}}\sum_n \mathrm{e}^{\mathrm{i}(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}_n},$$

where fA is the scattered wave amplitude from a single atom and $\mathbf{k}$ is the scattered wavevector (assumed the same for all atoms).

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the unit cell vectors of the lattice. A general reciprocal lattice vector takes the form

$$\mathbf{R} = s\mathbf{A} + t\mathbf{B} + u\mathbf{C},$$

where $\mathbf{A}\cdot\mathbf{a} = 2\pi$, $\mathbf{A}\cdot\mathbf{b} = \mathbf{A}\cdot\mathbf{c} = 0$, $\mathbf{B}\cdot\mathbf{b} = 2\pi$, $\mathbf{B}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{c} = 0$, $\mathbf{C}\cdot\mathbf{c} = 2\pi$, $\mathbf{C}\cdot\mathbf{a} = \mathbf{C}\cdot\mathbf{b} = 0$ and s, t, and u are integers. Show that, for atom locations

$$\mathbf{r}_n = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

where α, β, γ are integers, diffraction maxima will be observed at D when $\mathbf{k}_0 - \mathbf{k}$ is a reciprocal lattice vector.

Define a *Brillouin zone* and demonstrate that all wavevectors which start from the origin and satisfy the condition for a maximum will terminate on a Brillouin zone boundary.

A two-dimensional crystal contains a single type of atom. The atomic spacing is a along one axis of the unit cell, b along the second axis, and the angle between the axes is α. Find the reciprocal lattice for this crystal and draw a labelled diagram showing the first Brillouin zone.