Chapter 8

LAMINAR BOUNDARY LAYER FLOWS

8.1 Boundary Layer Flow

In this chapter, we consider flows near solid surfaces known as *boundary layer flows*. One way of describing these flows is in terms of vorticity dynamics, i.e., generation, diffusion, convection, and intensification of vorticity. The presence of vorticity distinguishes boundary layer flows from potential flows, which are free of vorticity.

In two-dimensional flow along the xy-plane, the vorticity is given by

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{u} = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \mathbf{e}_k \, , \tag{8.1}$$

and is a measure of rotation in the fluid. As discussed in Chapter 6, vorticity is generated at solid boundaries. For example, if $u_x = u_x(x, y)$ and the plane $y=0$ corresponds to an impermeable wall, then along this wall $u_y=0$ and $\partial u_y/\partial x=0$. Due to the no-slip boundary condition, $\partial u_x/\partial y$ is non-zero, and thus vorticity is generated according to

$$\boldsymbol{\omega} = -\frac{\partial u_x}{\partial y} \, \mathbf{e}_k \, . \tag{8.2}$$

Vorticity diffuses away from the generator wall at a rate of ($\nu \nabla^2 \boldsymbol{\omega}$), and competes with convection at a rate of ($\mathbf{u} \cdot \nabla \boldsymbol{\omega}$) (Fig. 8.1). Due to the effects of convection, the vorticity is confined within a parabolic-like envelope which is commonly known as *boundary layer*. Therefore, the area away from the solid wall remains free of vorticity.

The line separating boundary-layer and potential flows, i.e., the line where the velocity changes from a parabolic to a flat profile, is defined by the orbit of vorticity "particles" generated at a solid surface and diffused away to a *penetration* or *boundary layer thickness*, $\delta(x)$. As already discussed in Section 7.3, along the edge

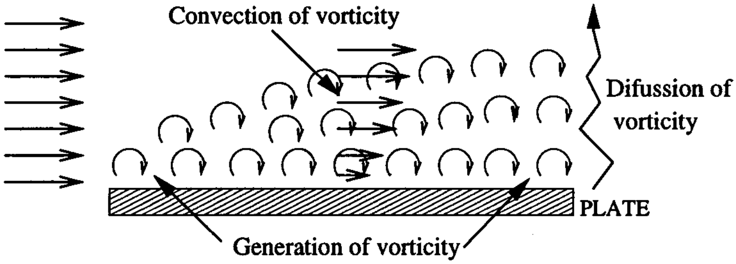


Figure 8.1. *Generation, diffusion and convection of vorticity in the vicinity of a solid wall.*

of the boundary layer, convection and diffusion of vorticity are of the same order of magnitude, i.e.,

$$V \frac{\partial \omega}{\partial x} \simeq k^2 \nu \frac{\partial^2 \omega}{\partial y^2} , \tag{8.3}$$

where k is a constant. Consequently,

$$\left[\frac{V}{x} \right] \simeq k^2 \left[\frac{\nu}{\delta^2(x)} \right] , \tag{8.4}$$

where x is the distance from the leading edge of the plate. Therefore, the expression

$$\delta(x) = k \sqrt{\frac{\nu x}{V}} , \tag{8.5}$$

provides an order of magnitude estimate for the boundary layer thickness.

Consider now the flow past a submerged body, as shown in Fig. 8.2. Across the boundary layer, the velocity increases from zero– due to the no-slip boundary condition– to the finite value of the free stream flow. The thickness of the boundary layer, $\delta(x)$, is a function of the distance from the leading edge of the body, and depends on the local Reynolds number, $Re \equiv \rho V x/\eta$; $\delta(x)$ can be infinitesimal, finite or practically infinite. When $Re \ll 1$ (which leads to creeping flow), the distance $\delta(x)$ is practically infinite. In this case, the solution to the Navier-Stokes equations for creeping flow (discussed in Chapter 10) holds uniformly over the entire flow area. For $1 \ll Re < 10^4$, $\delta(x)$ is small but finite, i.e., $\delta(x)/L \ll 1$. For higher Reynolds numbers, the flow becomes turbulent leading to a turbulent boundary layer. Under certain flow conditions, the boundary layer flow detaches from the

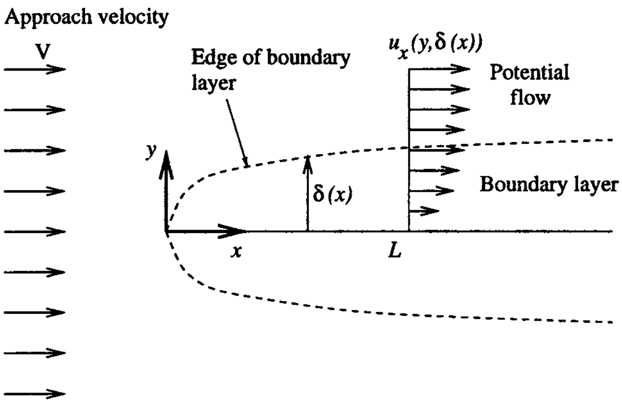


Figure 8.2. *Boundary layer and potential flow regions around a plate.*

solid surface, resulting in shedding of vorticity that eventually accumulates into periodically spaced traveling vortices that constitute the wake.

From the physical point of view, the boundary layer thickness $\delta(x)$ defines the region where the effect of diffusion of vorticity away from the generating solid surface competes with convection from bulk motion. A rough estimate of the thickness $\delta(x)$ is provided by Eq. (8.5). The presence of vorticity along and across the boundary layer is indicated in the schematic of Fig. 8.1. As discussed in Chapter 7, from a mathematical point of view, the solution within the boundary layer is an inner solution to the Navier-Stokes equations which satisfies the no-slip boundary condition, but not the potential velocity profile away from the body.

8.2 Boundary Layer Equations

Boundary layer flow of Newtonian fluids can be studied by means of the Navier-Stokes equations. However, the characteristics of the flow suggest the use of simplified governing equations. Indeed, using order of magnitude analysis, a more simplified set of equations known as the *boundary layer equations* [1], can be developed. In reference to Fig. 8.2, the Navier-Stokes equations are made dimensionless by means of characteristic quantities that bring the involved terms to comparable order of magnitude:

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{L} Re^{1/2},$$

$$u_x^* = \frac{u_x}{V}\,, \qquad u_y^* = \frac{u_y}{V} Re^{1/2}\,,$$
$$\tau_{ij}^* = \frac{\tau_{ij}}{\eta \frac{V}{\delta}}\,, \qquad p^* = \frac{p}{\rho V^2}\,,$$

where $Re \equiv VL/\nu$ is the Reynolds number. For steady flow, the resulting dimensionless equations are:

$$\frac{\partial u_x^*}{\partial x^*} + \frac{\partial u_y^*}{\partial y^*} = 0\,; \tag{8.6}$$

$$\left(u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_x^*}{\partial y^*} \right) = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u_x^*}{\partial y^{*2}} + \frac{1}{Re} \frac{\partial^2 u_x^*}{\partial x^{*2}}\,; \tag{8.7}$$

$$\frac{1}{Re} \left(u_x^* \frac{\partial u_y^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*} \right) = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re} + \frac{\partial^2 u_y^*}{\partial y^{*2}} + \frac{1}{Re} \frac{\partial^2 u_y^*}{\partial x^{*2}}\,. \tag{8.8}$$

If $Re \gg 1$, these equations reduce to

$$\frac{\partial u_x^*}{\partial x^*} + \frac{\partial u_y^*}{\partial y^*} = 0\,, \tag{8.9}$$

$$u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_x^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u_x^*}{\partial y^{*2}}\,, \tag{8.10}$$

and

$$p^* = p^*(x^*)\,. \tag{8.11}$$

The appropriate, dimensionless boundary conditions to Eqs. (8.9) to (8.11) are:

at $y^* = 0$, $\quad u_x^* = u_y^* = 0$ (no-slip);

at $y^* = 1$, $\quad u_x^* = 1$, $\quad \dfrac{\partial u_x^*}{\partial y^*} = 0$ (continuity of velocity and stress);

at $x^* = 0$, $\quad u_x^* = u_y^* = 0$ (stagnation point).

The pressure gradient, dp^*/dx^*, is identical to that of the outer, potential flow, $(dp^*/dx^*)_p$,

$$\frac{dp^*}{dx^*} = \left(\frac{dp^*}{dx^*} \right)_p = \left(-u_x^* \frac{du_x^*}{dx^*} \right)_p = \begin{cases} 0, & \text{slender body} \\ \text{known}, & \text{non} - \text{slender body} \end{cases} \tag{8.12}$$

Thus, the only unknowns in Eqs. (8.9) and (8.10) are the two velocity components, u_x^* and u_y^*. The latter is eliminated by means of the continuity equation,

$$u_y^* = -\int_0^{y^*} \frac{\partial u_x^*}{\partial x^*}\, dy^*\,, \tag{8.13}$$

leading to a single equation,

$$u_x^* \frac{\partial u_x^*}{\partial x^*} + \frac{\partial u_x^*}{\partial y^*} \left(- \int_0^{y^*} \frac{\partial u_x^*}{\partial x^*} dy^* \right) = - \frac{dp^*}{dx^*} + \frac{\partial^2 u_x^*}{\partial y^{*2}} \; . \tag{8.14}$$

The corresponding dimensional forms of the boundary layer equations for laminar flow are

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \; , \tag{8.15}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y} \; , \tag{8.16}$$

and

$$p = p(x) \; . \tag{8.17}$$

These constitute a set of non-linear parabolic equations for which an exact, closed-form solution is not possible. Blasius [2] developed an approximate similarity solution for flow past a flat plate, i.e., for $dp^*/dx^* = 0$. He introduced a dimensionless stream function, which, of course, satisfies the dimensionless continuity Eq. (8.9), of the form

$$\psi^* \equiv \frac{\psi}{\sqrt{\nu V x}} = f(\xi) \; , \tag{8.18}$$

where ξ is a similarity coordinate variable, defined as

$$\xi = y \sqrt{\frac{V}{\nu x}} \; .$$

By recognizing that an estimate of the boundary layer thickness is

$$\delta(x) \approx \sqrt{\frac{\nu x}{V}} \; ,$$

the variable ξ scales appropriately the coordinate across the thickness of the boundary layer,

$$\xi \; = \; \frac{y}{\delta(x)} \; . \tag{8.19}$$

From the definition of the stream function (Chapter 2),

$$u_x \; = \; \frac{\partial \psi}{\partial y} = V f' \; , \tag{8.20}$$

and

$$u_y = -\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu V}{x}}\;(\xi f' - f)\;. \tag{8.21}$$

Substitution of Eqs. (8.20) and (8.21) in Eq. (8.16) leads to the *Blasius equation*,

$$2\frac{d^3 f}{d\xi^3} + f\frac{d^2 f}{d\xi^2} = 0\,. \tag{8.22}$$

This is a nonlinear ordinary differential equation subject to the boundary conditions

$$f(0) = f'(0) = 0 \tag{8.23}$$

and

$$f'(\xi \to \infty) = 1\,. \tag{8.24}$$

In order to avoid the infinite domain of the two-point boundary value problem, Blasius solved Eq. (8.22) by transforming it to an equivalent forward numerical integration scheme of ordinary differential equations from ξ=0 up to a point ξ_∞ where the outer-edge boundary condition (8.24) is satisfied. These numerical results are tabulated in Table 8.1. In Fig. 8.3, the dimensionless velocity u_x/V is plotted versus the similarity variable ξ=$y/\delta(x)$=$y/(\nu x/V)^{1/2}$. Several other useful quantities can be calculated by means of the results given in Table 8.1, such as:

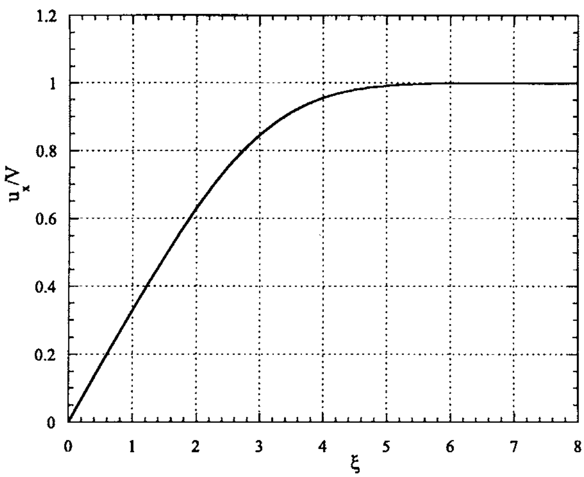


Figure 8.3. *Solution of the Blasius equation*

ξ	f	f'	f''
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.3320600
0.2500000	1.0376875E-02	8.3006032E-02	0.3319165
0.5000000	4.1494049E-02	0.1658866	0.3309136
0.7500000	9.3284436E-02	0.2483208	0.3282084
1.000000	0.1655748	0.3297825	0.3230098
1.250000	0.2580366	0.4095603	0.3146356
1.500000	0.3701439	0.4867927	0.3025829
1.750000	0.5011419	0.5605230	0.2866015
2.000000	0.6500322	0.6297698	0.2667536
2.250000	0.8155764	0.6936100	0.2434452
2.500000	0.9963216	0.7512640	0.2174131
2.750000	1.190646	0.8021722	0.1896628
3.000000	1.396821	0.8460487	0.1613615
3.250000	1.613085	0.8829061	0.1337038
3.500000	1.837715	0.9130443	0.1077739
3.750000	2.069094	0.9370083	8.4430709E-02
4.000000	2.305766	0.9555216	6.4236179E-02
4.250000	2.546470	0.9694083	4.7435224E-02
4.500000	2.790157	0.9795170	3.3984236E-02
4.750000	3.035983	0.9866555	2.3614302E-02
5.000000	3.283299	0.9915444	1.5911143E-02
5.250000	3.531620	0.9947910	1.0394325E-02
5.500000	3.780600	0.9968815	6.5830108E-03
5.750000	4.029997	0.9981864	4.0417481E-03
6.000000	4.279651	0.9989761	2.4056220E-03
6.250000	4.529459	0.9994394	1.3880555E-03
6.500000	4.779355	0.9997029	7.7646959E-04
6.750000	5.029300	0.9998482	4.2111927E-04
7.000000	5.279273	0.9999259	2.2145297E-04
7.250000	5.529260	0.9999662	1.1292604E-04
7.500000	5.779254	0.9999865	5.5846038E-05
7.750000	6.029253	0.9999964	2.6787688E-05
8.000000	6.279252	1.000001	1.2465006E-05

Table 8.1. *Solution of the Blasius equation.*

(a) Boundary layer thickness, $\delta(x)$
This is defined as the location where $u_x/V = u_x^* = 0.99$. From Fig. 8.3 (or Table 8.1), this occurs at $\xi \approx 5$. Hence,

$$\delta(x) \;=\; y(u_x^* = 0.99) \;=\; 5\sqrt{\frac{\nu x}{V}}\,. \tag{8.25}$$

(b) Wall shear stress, τ_w, and drag force, F_D
The shear stress at the plate is

$$\tau_w \;=\; \eta \frac{\partial u_x}{\partial y}\bigg|_{y=0} \;=\; \eta\sqrt{\frac{V^3}{\nu x}} f''(0) \;=\; 0.332\,(\rho V^2)\,\sqrt{\frac{\nu}{V x}}\,. \tag{8.26}$$

Therefore, the *drag force* on the plate per unit width is

$$F_D \;=\; \int_0^L \tau_w \, dx \;=\; 0.664\,\sqrt{V^3 \rho \eta L}\,. \tag{8.27}$$

The net normal force is zero. For the *drag coefficient*, C_D, defined by

$$C_D \equiv \frac{F_D}{\frac{\rho V^2}{2} L}\,,$$

we get

$$C_D \;=\; \frac{1.328}{\sqrt{\frac{VL}{\nu}}}\,. \tag{8.28}$$

Notice that the local shear stress breaks down at $x=0$, where τ_w is singular, i.e., $\tau_w \rightarrow \infty$ as $x \rightarrow 0$. Actually, the formula does not apply there because the potential flow approximation is not valid near $x=0$. Nevertheless, the singularity is *integrable*, i.e., the drag F_D is finite.

(c) The small normal velocity component, u_y
At $\xi=5$,

$$u_y \;=\; -\frac{\partial \psi}{\partial x} \;=\; \frac{1}{2}\sqrt{\frac{V\nu}{x}}\,(\xi f' - f) \;=\; 0.837\,\sqrt{\frac{V\nu}{x}}\,. \tag{8.29}$$

(d) Transition to turbulent boundary layer
Transition to turbulent boundary layer occurs at $Re = Vx/\nu \simeq$112,000, or at distance x downstream from the leading edge, given by

$$x \;=\; \frac{112,000\,\nu}{V}\,. \tag{8.30}$$

The dimensionless boundary layer equations, and their solution are independent of the Reynolds number. *Thus, all boundary layer flows in similar geometries are dynamically similar. The Reynolds number is only a scaling factor of the boundary layer thickness and the associated variables.*

8.3 Approximate Momentum Integral Theory

Reasonably accurate solutions to boundary layer flows can be obtained from macroscopic mass and momentum balances through the use of finite control volumes. This method was first introduced by von Karman [3], and is highlighted below for boundary layer over a flat plate (Fig. 8.4).

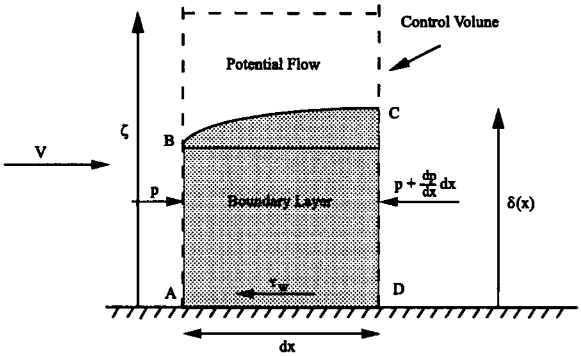


Figure 8.4. *Derivation of von Karman's approximate equations for boundary layer flow over a flat plate.*

In von Karman's method, the flow is integrated across the thickness of the layer. To account for the development of the boundary layer, and to ensure that its thickness is confined within the limits of integration, the control volume is selected so that its size, ζ, is larger than the expected thickness of the layer. Integrating from 0 to ζ gives

$$\int_0^\zeta \left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) dy = \int_0^\zeta -\frac{1}{\rho}\frac{\partial p}{\partial x}\, dy + \int_0^\zeta \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y}\, dy \, .$$

From the continuity equation,

$$u_y = -\int_0^y \frac{\partial u_x}{\partial x} dy \, .$$

For steady flow, the pressure gradient is

$$\frac{1}{\rho}\frac{dp}{dx} = -V\frac{dV}{dx} \ .$$

By substitution, we get

$$\int_0^\zeta u_x \frac{\partial u_x}{\partial x} dy - \int_0^\zeta \left(\int_0^y \frac{\partial u_x}{\partial x} dy \right) \frac{\partial u_x}{\partial y} dy = \int_0^\zeta V \frac{\partial V}{\partial x} dy + \int_0^\zeta \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} dy \ .$$

Defining now the new variables A and B, such that

$$dB = \frac{\partial u_x}{\partial y} dy \qquad \Longrightarrow \qquad B = u_x \ ,$$

and

$$A = \int_0^y \frac{\partial u_x}{\partial x} dy \qquad \Longrightarrow \qquad dA = \frac{\partial u_x}{\partial x} dy \ ,$$

the second term in the momentum equation can be written as

$$\int_0^\zeta A \, dB = AB|_0^\zeta - \int_0^\zeta B dA = V \int_0^\zeta \frac{\partial u_x}{\partial x} \, dy - \int_o^\zeta u_x \frac{\partial u_x}{\partial x} dy \ .$$

Finally, since ζ is independent of x, by integrating and rearranging, the integral form of the momentum equation becomes

$$\frac{\partial}{\partial x} V^2 \int_0^\zeta \left[\frac{u_x}{V}\left(1 - \frac{u_x}{V}\right)\right] dy + V\frac{\partial V}{\partial x}\int_0^\zeta \left(1 - \frac{u_x}{V}\right) dy = \frac{\tau_w}{\rho} \ , \tag{8.31}$$

where τ_w is the shear stress at the wall.

The above integral equation can be simplified further to

$$\frac{\partial}{\partial x}\left[V^2 \, \delta_2(x)\right] + \delta_1(x) \, V\frac{\partial V}{\partial x} = \frac{\tau_w}{\rho} \ , \tag{8.32}$$

where $\delta_1(x)$ is the *displacement thickness*, and $\delta_2(x)$ is the *momentum thickness.* Equation (8.32) is known as the *momentum-integral equation*, or as *von Karman's integral equation.*

The displacement thickness, $\delta_1(x)$, is associated with the reduction in the mass flow rate per unit depth in the boundary layer, as a result of the velocity slow-down $(V - u_x)$, i.e.,

$$\dot{m}_r = \rho \int_0^\delta (V - u_x) \, dy \ .$$

When the rate of "mass loss", $\dot{m}_r$, is expressed in terms of an equivalent thickness δ_1, of uniform flow with the same mass flow rate, i.e., $\dot{m}_r = \rho\, V\, \delta_1$, we have

$$\dot{m} = \rho\, \delta_1(x)\, V = \rho \int_0^\zeta (V - u_x)\, dy\ ,$$

which leads to the definition of $\delta_1(x)$ as

$$\delta_1(x) = \int_0^\zeta \left(1 - \frac{u_x}{V}\right) dy\ . \tag{8.33}$$

The *momentum thickness*, $\delta_2(x)$, is related to the deficiency in the momentum per unit depth, $\dot{J}_D$, associated with the slow-down of the velocity within the boundary layer,

$$\dot{J}_D = \rho \int_0^\delta u\,(V - u)\ dy\ .$$

This momentum deficiency produces a net force along the direction of flow. Thus, $\delta_2(x)$ is defined as the thickness of uniform flow that carries the same momentum, i.e., $\dot{J}_D = \rho\, \delta_2\, V^2$. Therefore,

$$\rho\, \delta_2(x)\, V^2 \ = \ \rho \int_0^\zeta u_x\,(V - u_x)\, dy$$

or

$$\delta_2(x) \ = \ \int_0^\zeta \left[\frac{u_x}{V}\left(1 - \frac{u_x}{V}\right)\right] dy\ . \tag{8.34}$$

Since the boundary layer thickness increases in the direction of flow, both δ_1 and δ_2 are functions of x. The approximate integral momentum equation (8.32) is derived without any assumption concerning the nature of the flow. Therefore, it applies to both laminar and turbulent flows.

Example 8.3.1. Von Karman's boundary layer solution

Consider the velocity profile

$$\frac{u_x}{V} \ = u_x^* = a\xi^3 + b\xi^2 + c\xi + d \quad \text{with} \quad \xi = \frac{y}{\delta}\ . \tag{8.35}$$

$$\begin{aligned}
&\text{At} \quad \xi = 0\,, \quad u_x^* = 0\,; \quad \text{therefore, } d = 0\,.\\
&\text{At} \quad \xi = 1\,, \quad \frac{\partial u_x^*}{\partial \xi} = 0\,; \quad \text{therefore, } 3a + 2b + c = 0\,.\\
&\text{At} \quad \xi = 1\,, \quad u_x^* = 1\,; \quad \text{therefore, } a + b + c = 1\,.
\end{aligned}$$

An additional boundary condition is obtained at y=0 by satisfying the momentum equation there

$$\left[u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_x^*}{\partial y^*}\right]_{y^*=0} = -\left.\frac{dp^*}{dx^*}\right|_{y^*=0} + \left.\frac{\partial^2 u_x^*}{\partial y^{*2}}\right|_{y^*=0} . \tag{8.36}$$

Since u_x^*=u_y^*=0 and dp^*/dx^*=0 (for flow past a flat plate),

$$\left.\frac{\partial^2 u_x^*}{\partial y^{*2}}\right|_{y^*=0} = 0 . \tag{8.37}$$

Therefore,

$$\text{At} \quad \xi = 0 , \quad \frac{\partial^2 u_x^*}{\partial \xi^2} = 0 \quad \text{and} \quad b = 0 . \tag{8.38}$$

The four boundary conditions determine the unknown coefficients as

$$b = d = 0 , \quad c = 3/2 \quad \text{and} \quad a = -1/2 .$$

Therefore, the admissible velocity profile is

$$u_x^* = \frac{u_x}{V} = \frac{3}{2}\,\xi - \frac{1}{2}\,\xi^3 . \tag{8.39}$$

Equation (8.39) is substituted in von Karman's equation, Eq. (8.32), to yield

$$\tau_w = \rho V^2 \frac{d\delta}{dx} \int_0^1 \left(1 - \frac{u_x}{V}\right) \frac{u_x}{V}\, d\xi = 0.139\, \rho V^2\, \frac{d\delta}{dx} . \tag{8.40}$$

Also,

$$\tau_w = \eta \frac{du_x}{dy} = \eta \frac{V}{\delta} \left.\frac{du_x^*}{d\xi}\right|_{\xi=0} = \frac{3}{2}\eta\, \frac{V}{\delta} . \tag{8.41}$$

Equation (8.40) then becomes

$$\delta\; d\delta = 8.791 \frac{\eta}{\rho V} dx \quad \text{with} \quad \delta(x = 0) = 0 , \tag{8.42}$$

which is solved for the boundary layer thickness,

$$\delta(x) = 4.646 \sqrt{\frac{\nu x}{V}} . \tag{8.43}$$

The exact value given by Eq. (8.25), is, therefore, represented reasonably well by the approximate expression shown in Eq. (8.43) with a difference of less than 10%. A similar analysis can be repeated by means of exponential, or other similar functional forms for the velocity distribution. The error between the exact and the approximate solutions depends on the approximating velocity profile. □

Example 8.3.2

Another approximate qualitative expression for the thickness $\delta(x)$ can be derived by approximating the velocity profile throughout the flow as:

$$u_x(x,y) = V\left(1 - e^{-ay}\right), \tag{8.44}$$

where a is a function of x, and accounts for the growth of the boundary layer. In this case, the match with the potential flow $u_x = V$, occurs at a large distance away from the boundary layer, i.e., as $y \to \infty$. For this velocity profile,

$$\delta_2(x) = \int_0^\infty \left[\frac{u_x}{V}\left(1 - \frac{u_x}{V}\right)\right] dy = \int_0^\infty \exp^{-ay}\left(1 - \exp^{-ay}\right) dy = \frac{1}{2a}. \tag{8.45}$$

The wall shear stress is given by

$$\tau_w = \eta \left.\frac{\partial u_x}{\partial x}\right|_{y=0} = \eta a V. \tag{8.46}$$

Substitution of Eqs. (8.45) and (8.46) into Eq. (8.32), for dp/dx=0, yields

$$\frac{da}{a^3} = -\frac{2\eta}{\rho V}\, dx, \tag{8.47}$$

which is integrated to

$$a = \left(\frac{V}{4\nu x}\right)^{1/2}. \tag{8.48}$$

Therefore, the velocity profile becomes

$$u_x(x,y) = V\left[1 - \exp\left(-y\sqrt{\frac{V}{4\nu x}}\right)\right]. \tag{8.49}$$

In practice, the boundary layer thickness is taken as the distance where u_x=0.99V. Therefore,

$$\delta(x) = \ln 100\sqrt{\frac{\nu x}{V}} = 4.54\sqrt{\frac{4\nu x}{V}}. \tag{8.50}$$

The velocity profile can also be written as

$$u_x(x,y) = V\left[1 - \exp\left(-4.54\frac{y}{\delta}\right)\right] . \tag{8.51}$$

The vorticity distribution is

$$\omega(x,y) = \frac{\partial u_x}{\partial y} = 4.54\,\frac{V}{\delta}\exp\left(-4.54\,\frac{y}{\delta}\right) . \tag{8.52}$$

At the edge of the boundary layer, i.e., at $y{=}\delta$,

$$\omega(x,\delta) = 4.54\,\frac{V}{\delta}e^{-4.54} = 0.0454\,\frac{V}{\delta} . \tag{8.53}$$

Therefore, the vorticity decays from its boundary value,

$$\omega(x,0) = 4.54\,\frac{V}{\delta} , \tag{8.54}$$

to the minimum vorticity of the boundary layer,

$$\omega(x,\delta) = 0.0454\,\frac{V}{\delta} , \tag{8.55}$$

and, further, quite sharply to

$$\omega(x,2\delta) \simeq 10^{-3}\,\frac{V}{\delta} , \tag{8.56}$$

at a distance just twice the boundary layer thickness. □

Example 8.3.3. Blasius and Sakiades boundary layers

Consider the two flow configurations shown in Fig. 8.5, studied by Blasius [2] and Sakiades [4]. In light of the previous discussions, the boundary conditions for the two flows are:

Blasius flow

At $y = 0,\ u_x = u_y = 0$

As $y \to \infty,\ u_x \to V$

As $\left.\begin{array}{l} x \to 0 \\ y \neq 0 \end{array}\right\} u_x \to V$

Sakiades flow

At $y = 0,\ u_x = V,\ u_y = 0$

As $y \to \infty,\ u_x \to 0$

As $\left.\begin{array}{l} x \to 0 \\ y \neq 0 \end{array}\right\} u_x \to 0$

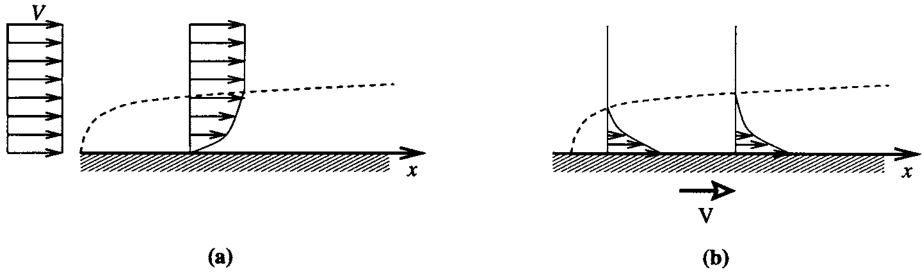


Figure 8.5. *(a) Blasius boundary layers arise when liquid streams overtake stationary bodies; (b) Sakiades boundary layers arise when bodies travel in stationary liquids.*

Although the boundary conditions in the two flows are different, the boundary layer equations with no streamwise pressure gradient apply to both cases:

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \nu \frac{\partial^2 u_x}{\partial y^2} , \tag{8.57}$$

and

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 . \tag{8.58}$$

By means of the stream function, ψ,

$$u_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y = -\frac{\partial \psi}{\partial x} , \tag{8.59}$$

and the system of Eqs. (8.57) and (8.58) reduces to

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^2 \psi}{\partial y^3} . \tag{8.60}$$

In terms of the stream function, the boundary conditions are:

Blasius flow

At $\quad y = 0 , \quad \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \psi}{\partial x} = 0$

As $\quad y \to \infty , \quad \psi \to V y$

As $\quad \left. \begin{array}{l} x \to 0 \\ y \neq 0 \end{array} \right\} \quad \psi \to V y$

Sakiades flow

At $\quad y = 0 , \quad \psi = V y , \quad \dfrac{\partial \psi}{\partial x} = 0$

As $\quad y \to \infty , \quad \psi \to \text{constant} \quad (0, \text{say})$

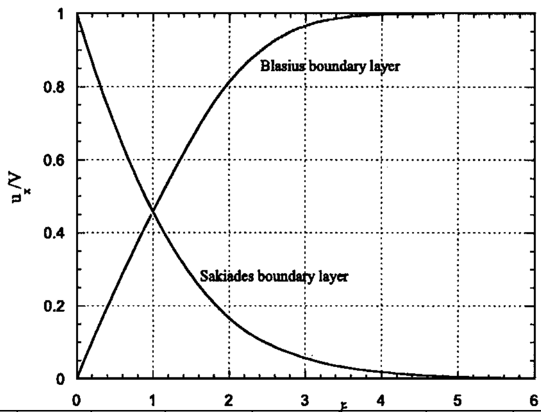
As $\quad \left. \begin{array}{l} x \to 0 \\ y \neq 0 \end{array} \right\} \quad \psi \to \text{constant} \quad (0, \text{say})$

Since there is no length scale in this problem, the solution must be independent of the unit chosen for length. By dimensional analysis, it follows that

$$\frac{\psi}{\sqrt{2\nu V x}} = f\left(y\sqrt{\frac{V}{2\nu x}} \right) . \tag{8.61}$$

Note that a factor of 2 is included for convenience. The most important aspects of the two boundary layer solutions are:

Velocity and boundary layer thickness



ξ	0	1	2	3	3.5	4.3	4.9	5.4	6
u_x/V, Blasius	0	0.46	0.82	0.97	0.99	0.999	.9999	.99999	.999999
u_x/V, Sakiades	1	0.45	0.16	0.055	0.031	0.013	.0063	.0036	.0018

Wall shear stress

By means of

$$\tau_w = \eta \frac{\partial u_x}{\partial y}\bigg|_{y=0} = \eta\sqrt{\frac{V^3}{2\nu x}} f''(0) ,$$

we find that

$$\tau_w^B = 0.332(\rho V^2)\sqrt{\frac{\nu}{Vx}} \qquad \text{and} \qquad \tau_w^S = -0.444\rho V^2\sqrt{\frac{\nu}{Vx}} , \tag{8.62}$$

where superscripts B and S denote the Blasius and the Sakiades solutions, respectively.

Drag on a finite length

The drag on a finite length, ℓ, per unit width is given by

$$F_D = \int_0^\ell \tau_w \, dx ,$$

which gives

$$F_D^B = 0.664\sqrt{V^3\rho\eta\ell} \qquad \text{and} \qquad F_D^S = -0.888\sqrt{V^3\rho\eta\ell}\,. \tag{8.63}$$

Traverse velocity

This is given by

$$u_y = \sqrt{\frac{\nu V}{2x}}\,[\xi f'(\xi) - f(\xi)]\ ,$$

which yields

$$\frac{u_y^B}{V} \to 0.837\sqrt{\frac{\nu}{Vx}} \qquad \text{and} \qquad \frac{u_y^S}{V} = -0.808\sqrt{\frac{\nu}{Vx}}\,. \tag{8.64}$$

Therefore, the laminar flow drag force on a flat surface moving through still fluid with velocity V is about 34% greater than the drag force on the same surface due to fluid flowing past it with velocity V. In the former case, there is a drift velocity *towards* the plate because the fluid is *accelerated*, while in the latter case this situation is reversed. □

8.4 Boundary Layers within Accelerating Potential Flow

The boundary layer thickness is defined by the competition between convection, which tends to confine vorticity close to its generating source, and by diffusion that drives to vorticity uniformity away from the solid surface. Besides these effects, due to the dominant velocity component, vorticity penetration is enhanced by a small vertical velocity component away from the boundary, as in the case of the Blasius boundary layer, and is inhibited by a small vertical velocity component towards the boundary, as in the case of the Sakiades boundary layer. At the outer part of the boundary layer, the vertical velocity component is related to the potential velocity profile, $V(x)$, by the continuity equation,

$$u_y = -\int_0^y \frac{\partial u_x}{\partial x} dy \simeq -\int_0^y \frac{\partial V(x)}{\partial x} dy\ . \tag{8.65}$$

Equation (8.65) suggests that an accelerating potential flow, of $\partial V/\partial x > 0$, induces a small vertical velocity component towards the boundary, which in turn confines vorticity from penetrating away, and therefore reduces the thickness of the boundary layer. The opposite is true for a decelerating potential flow, of $\partial V/\partial x < 0$, that induces a small vertical velocity component away from the boundary, and therefore increases the boundary layer thickness.

Bernoulli's equation along a potential streamline of arc length s takes the form

$$V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s}\,. \tag{8.66}$$

Thus, an accelerating potential velocity results in a negative pressure gradient, which, according to the boundary layer momentum equation, tends to diminish the variation of the velocity across the boundary layer; therefore, it tends to decrease the boundary layer thickness. The opposite is true for a decelerating potential velocity.

Falkner and Skan [5], extended Blasius boundary layer analysis to cases with an external potential velocity field of the type

$$V = c\,x^m\,, \tag{8.67}$$

where c and m are positive constants. The stream function,

$$\psi = (\nu V x)^{1/2} f(\eta)\,, \tag{8.68}$$

where η is a similarity variable defined by

$$\eta = (V/\nu x)^{1/2} y\,, \tag{8.69}$$

transforms the momentum equation to the ordinary differential equation,

$$m f'^2 - \frac{1}{2}(m+1) f f'' = m + f'''\,. \tag{8.70}$$

Note that the above equation reduces to Eq. (8.22) in the limit of m=0. In stagnation flow, where a jet impinges on a vertical wall (m=1), Eq. (8.70) becomes

$$f'^2 - f f'' = 1 + f'''\,. \tag{8.71}$$

The corresponding boundary conditions are

$$f(0) = f'(0) = 0\,, \tag{8.72}$$

and

$$f'(\eta \to \infty) = 1\,. \tag{8.73}$$

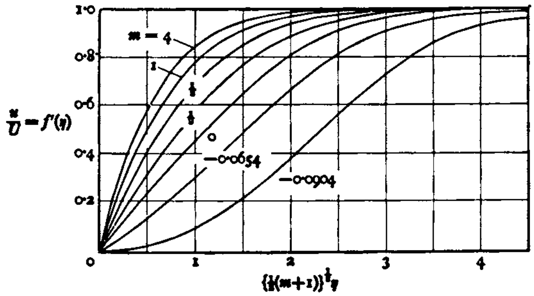


Figure 8.6. *Similarity distributions of velocity across the boundary layer in an external potential field of velocity, $V = cx^m$. [Taken from Ref. 6, by permission.]*

The numerical solution of Eqs. (8.71) to (8.73) is shown in Fig. 8.6, where the velocity distributions at different values of $\eta(x,y)=(V/\nu x)^{1/2}y$ are given. The boundary layer thickness is defined so that

$$\frac{u_x}{V}\left(\eta = \delta^{(m)}\sqrt{\frac{V}{\nu x}}\right) = 0.99\ . \tag{8.74}$$

Other useful quantities, such as the boundary shear stress,

$$\tau_w^{(m)} = \eta\,\frac{\partial u_x^{(m)}}{\partial y}\bigg|_{y=0} = \rho\left(\frac{\nu V^3}{x}\right)^{1/2}\left(f_0^{(m)}\right)''\ , \tag{8.75}$$

the vertical velocity component, and the total drag force are computed accordingly.

Example 8.4.1. Stagnation flow boundary layer

Consider the stagnation point flow shown in Fig. 8.7. The free stream velocity described by the stream function

$$\psi_p = k\,xy\ , \tag{8.76}$$

with velocity components

$$U = kx \qquad V = -ky\ , \tag{8.77}$$

impinges normal to the plate and forms a boundary layer of thickness $\delta(x)$. Since the potential velocity component, V, depends only on y and the other component, U, depends on x, the following form of the stream function is suggested (within the boundary layer)

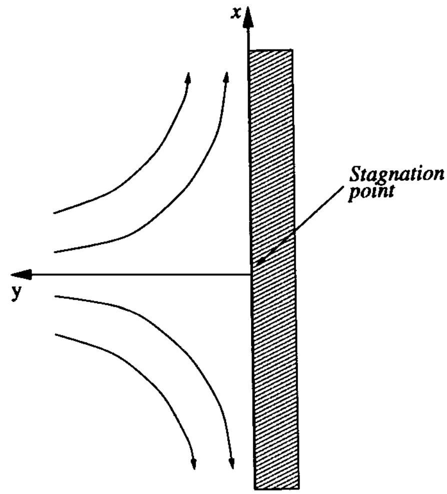


Figure 8.7. *Stagnation point flow.*

$$\psi = xf(y)\,. \tag{8.78}$$

The individual velocity components are

$$u_x = xf'(y) \quad \text{and} \quad u_y = -f(y)\,, \tag{8.79}$$

whereas the vorticity is given by

$$\omega = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = -xf''(y)\,. \tag{8.80}$$

Substituting these expressions in the boundary layer vorticity equation, we get

$$u_x\frac{\partial \omega}{\partial x} + u_y\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right)\,, \tag{8.81}$$

which leads to the ordinary differential equation,

$$-f'f'' + ff''' + \nu f'''' = 0\,, \tag{8.82}$$

with boundary conditions,

$$f(y=0) = f'(y=0) = 0\,, \tag{8.83}$$

and

$$f(y \to \infty) = k\,y\,. \tag{8.84}$$

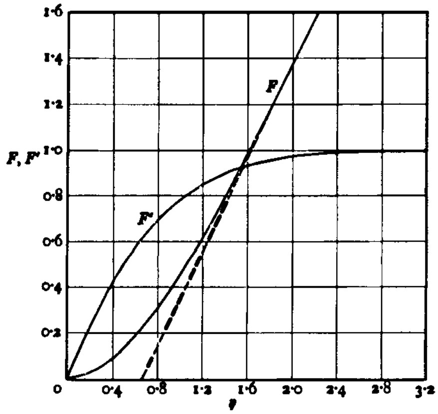


Figure 8.8. *Boundary layer results according to Eqs. (8.79) to (8.88). [Taken from Ref. 6, by permission.]*

By means of

$$y = \left(\frac{\nu}{k}\right)^{1/2} \eta \quad \text{and} \quad f(y) = (\nu k)^{1/2} F(\eta)\,, \tag{8.85}$$

Eqs. (8.82) to (8.84) are made dimensionless, as follows:

$$F'^2 - FF'' - F''' = 1\,, \tag{8.86}$$

$$F(\eta=0) = F'(\eta=0) = 0\,, \tag{8.87}$$

$$F(\eta \to \infty) = \eta\,. \tag{8.88}$$

Hiemenz solved these equations numerically [7]. The most important of his calculations are shown in Fig. 8.8. The resulting boundary layer thickness is

$$\delta(x) = 2.4\sqrt{\frac{\nu}{k}}\,, \tag{8.89}$$

which is independent of x. This is a consequence of the fact that the controlling velocity component of the potential flow V, is uniform over x, whereas the x-component varies linearly with x, hence, vorticity is convected downstream parallel to the wall within the boundary layer.

The corresponding axisymmetric case, that may occur when bodies of revolution move parallel to their axis of symmetry and form a stagnation point at the leading edge was treated by Homann [8]. Both the two-dimensional and the axisymmetric flows are special cases of Howarth's analysis of the general stagnation flow [9]. □

8.5 Flow over Non-Slender Planar Bodies

In flow past non-slender bodies, Eqs. (8.15) to (8.17) still apply. However, the Blasius exact solution does not apply, because

$$\frac{dp}{dx} = V(x)\frac{dV(x)}{dx} \neq 0\,, \tag{8.90}$$

due to the fact that, outside the boundary layer, the velocity $V(x)$ of the potential flow is deflected by the two-dimensional body, and made different from the approaching stream velocity, V. In this case, the solution is found iteratively; initially dp/dx is assumed zero, u_x is defined and δ is calculated from the corresponding von Karman equation which for non-slender bodies takes the form

$$\frac{d}{dx}\left[V^2(x)\int_0^\delta \frac{u_x}{V(x)}\left(1-\frac{u_x}{V(x)}\right)dy\right] + V(x)\frac{dV(x)}{dx}\int_0^\delta\left(1-\frac{u_x}{V(x)}\right)dy = \frac{\tau_w}{\rho}\,, \tag{8.91}$$

where

$$\frac{dp}{dx} = \rho\, V(x)\,\frac{dV(x)}{dx}\,. \tag{8.92}$$

Then dp/dx is calculated and the iteration is repeated until consecutive values of dp/dx do not differ much. In cases where the potential velocity is known (e.g., flow around a wedge-like, two-dimensional body), dp/dx is substituted directly in Eq. (8.91), and the iterative procedure is avoided.

8.6 Rotational Boundary Layers

A solid body moving with relative velocity V with respect to its surrounding liquid that cannot slip, generates an average vorticity, ω, of the order of $\omega \simeq V/\delta(x)$, that penetrates to a distance $\delta(x)$ under the combined action of convection and diffusion. Similarly, a disk in relative rotation Ω with respect to its surrounding liquid generates vorticity ω equal to twice its angular speed Ω that spreads in the vertical direction. Since the vorticity gradient in the azimuthal direction is zero, vorticity also spreads in the radial directions by convection and diffusion. At very low rotational speeds or Reynolds numbers, where centrifugal effects are negligible and Coriolis effects are dominant, a strictly azimuthal motion near the disk may be conserved. However, at high rotational speeds and Reynolds numbers, where the ratio of centrifugal/Coriolis effects is reversed, a strictly circular motion cannot be maintained and the fluid near the disk spirals outwards. Under these high Reynolds numbers, an axial motion towards the disk is induced, by virtue of mass conservation. This vertical velocity component confines the vorticity at the disk's surface within a finite distance and defines a rotational boundary layer. Briefly, the rotating disk operates as a centrifugal fan that receives fluid vertically and delivers it nearly radially.

For this class of problems, von Karman was the first to suggest that a solution of the form, $u_r{=}rf(z)$, $u_\theta{=}rg(z)$ and $u_z{=}h(z)$ is possible [2]. The boundary conditions are

$$u_r(z=0) = u_z(z=0) = 0\,; \tag{8.93}$$

$$u_\theta(z=0) = \Omega r\,; \tag{8.94}$$

$$u_r(z \to \infty) = u_z(z \to \infty) = 0\,. \tag{8.95}$$

The z-momentum equation becomes

$$\frac{\partial p}{\partial z} = \eta \frac{du_z}{dz} - \frac{\rho}{2} u_z^2 + c\,, \tag{8.96}$$

which suggests that

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \theta} = 0\,. \tag{8.97}$$

The remaining momentum equations, in the r- and θ-directions, become

$$\frac{u_r^2}{r^2} + u_z \frac{d}{dz}\left(\frac{u_r}{r}\right) - \frac{u_\theta^2}{r^2} = \nu \frac{d^2}{dz^2}\left(\frac{u_r}{r}\right)\,, \tag{8.98}$$

and

$$\frac{2u_r u_\theta}{r^2} + u_z \frac{d}{dz}\left(\frac{u_\theta}{r}\right) = \nu \frac{d^2}{dz^2}\left(\frac{u_\theta}{r}\right)\,, \tag{8.99}$$

respectively. Finally, the continuity equation simplifies to

$$\frac{2u_r}{r} + \frac{du_z}{dz} = 0\,. \tag{8.100}$$

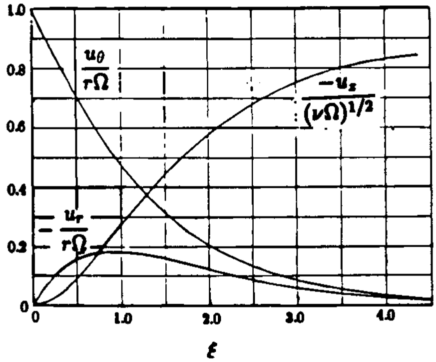


Figure 8.9. *Velocity components, according to Eqs. (8.102) and (8.107),over a fast rotating disk. [Taken from Ref. 10, by permission.]*

By introducing the transformation

$$z = \left(\frac{\nu}{\Omega}\right)^{1/2} \xi\,, \qquad \frac{u_\theta}{r} = \Omega\, g(\xi) \quad \text{and} \quad u_z = (\nu\Omega)^{1/2}\, h(\xi)\,, \tag{8.101}$$

the continuity equation is transformed to

$$\frac{u_r}{r} = \frac{\Omega}{2}\, h'(\xi)\,. \tag{8.102}$$

Equations (8.98) and (8.99) become

$$\frac{1}{4}h'^2 - \frac{1}{2}hh'' - g^2 = -\frac{1}{2}h'''\,, \tag{8.103}$$

and

$$-gh' + g'h = g''\,, \tag{8.104}$$

subject to

$$h(\xi = 0) = h'(\xi) = 0\,; \tag{8.105}$$

$$g(\xi = 0) = 1\,; \tag{8.106}$$

$$h'(\xi \to \infty) = g(\xi \to \infty) = 0\,. \tag{8.107}$$

Cochran solved the above equations numerically [9]. His main results are shown in Fig. 8.9. By defining the thickness of the boundary layer as the distance ξ where $u_\theta = 0.01\Omega r$, it turns out that the thickness is uniform,

$$\delta = 5.4\left(\frac{\nu}{\Omega}\right)^{1/2} = 5.4\left(\frac{\nu r}{V}\right)^{1/2}, \tag{8.108}$$

which is similar to a stagnation-type boundary layer thickness.

8.7 Problems

8.1. Water approaches an infinitely long and thin plate with uniform velocity.
(a) Determine the velocity distribution u_x in the boundary layer given that

$$u_x(x,y) \;=\; a(x)y^2 + b(x)y + c(x)\,.$$

(b) What is the flux of mass (per unit length of plate) across the boundary layer?
(c) Calculate the magnitude and the direction of the force needed to keep the plate in place.

8.2. In applying von Karman's momentum balance method to boundary layer flows, one is not restricted to piecewise differentiable approximations of the form

$$u_x^* = \begin{cases} f(\eta)\,, & 0 \le \eta \le 1 \\ V_\infty^*\,, & \eta > 1 \end{cases}$$

for the velocity distribution. The exact solution for the case of flow near a wall suddenly set in motion (derived in Chapter 6) suggests using the continuously differentiable velocity distribution

$$u_x^* \;=\; erf(\eta), \quad \eta \ge 0\,,$$

for the boundary layer flow past a flat plate. This distribution satisfies the following conditions:

$$\begin{aligned} u_x^* &= 0 \quad \text{at} \quad \eta = 0\,; \\ u_x^* &\to 1 \quad \text{as} \quad \eta \to \infty\,; \\ \left(\frac{\partial^2 u_x^*}{\partial y^{*2}}\right)_{\eta=0} &= \frac{dp^*}{dx^*} = 0\,. \end{aligned}$$

Derive the boundary layer thickness, the displacement thickness, the momentum thickness, the shear stress at the wall and the drag force over a length L. Compare with the results based on the traditional piecewise differentiable distributions.

8.3. Derive a formula for estimating how far downstream of a smooth, rounded inlet the parabolic velocity profile of plane Poiseuille flow becomes fully developed. Assume that the core flow in the entry region is rectilinear and irrotational, and that the velocity distribution in the boundary layers is self-similar:

$$\frac{u_x(x,y)}{U(x)} = f\left(\frac{y}{\delta(x)}\right) .$$

Use von Karman's integral momentum equation. Find the corresponding equation for pressure drop in the entry length.

8.4. Consider the *solid jet* flow induced by a continuous solid sheet emerging at constant velocity from a slit into fluid at rest.

(a) Justify the boundary-layer approximation for the flow.

(b) Show that the boundary conditions differ from those for flow past a flat plate, although the Blasius equation for the stream function applies.

(c) Employ von Karman's momentum equation to obtain approximate solutions for the local wall shear stress, the total drag on the two surfaces of the sheet, the displacement thickness and the momentum thickness, using for the velocity profile

i. a fourth-degree polynomial in $\eta \equiv y/a\,\sqrt{V/\nu x}$, and

ii. the complementary error function, erfc η = 1- erf η ,

where a is arbitrary constant. Compare the results with those for flow past a flat plate. (According to the exact solution for the solid jet the dimensionless local stress is 0.444, as compared with 0.332 for flow past a flat plate.)

8.5. *Laminar, incompressible, two-dimensional jet.*

(a) Justify the boundary-layer approximation for the flow caused by a fine sheet-jet emerging from slit into fluid at rest.

(b) Noting that the total momentum flux in the x-direction must be independent of distance x from the slit, i.e.,

$$\int_{-\infty}^{\infty} \rho u_x^2 \, dy = J = \text{const.}$$

establish that the velocity can depend on position only through the dimensionless combination

$$\eta = \frac{y}{\alpha}\left(\frac{J}{\rho\nu^2 x^2}\right)^{1/3}$$

where α is an arbitrary constant.

(c) Introduce a dimensionless stream function $f(\eta)$ and choose the constant α in such a way that the boundary-layer equation reduces to the ordinary differential equation

$$f^{'''} + 2ff^{''} + 2f^{'2} = 0\,.$$

(d) Show that the solution of this equation is $\alpha\,\tanh\alpha\eta$ and evaluate the constant α.

(e) Calculate u_x and u_y and the total volumentric flow rate, Q, entrained (as a function of x).

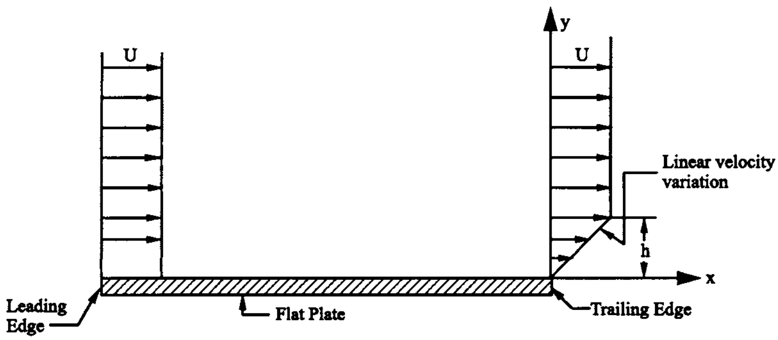


Figure 8.10. *Schematic of the flow in Problem 8.6.*

8.6. Consider two-dimensional boundary layer flow of water over one side of a flat plate. At the leading edge the velocity is uniform and equal to U. Downstream at the trailing edge, the velocity profile is as shown in Fig. 8.10.

(a) Find the shear force of the fluid on the plate by using an overall control volume approach.

(b) Sketch the velocity development from the leading to the trailing edge.

(c) Find an approximation for the normal velocity profile.

(d) Split the entire field into a boundary layer and a potential flow region (graphically.)

(e) Construct qualitative plots of the vorticity as a function of x at three distances from the plate: y=0, y=$h/2$ and y=$2h$.

8.7. For the general case of two-dimensional planar flow, Prandtl's boundary layer equations are [1]

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u_x}{\partial y^2} ,$$

and

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 .$$

Given the geometry of the problem and $U_0(x)$, the solution of these equations yields $u_x(x, y)$ and $u_y(x, y)$ inside a boundary layer. Consider the flow normal to the axis of a cylinder of arbitrary cross-section.

(a) Define x and y for this geometry. (Hint: consider flow along a flat plate.)

(b) What does $U(x)$ represent?

(c) What does UdU/dx represent physically?

(d) What are the appropriate boundary conditions?

(e) Does this solution correctly predict the drag force on the cylinder? Explain your answer as quantitatively as possible.

8.8. *Boundary layer over wedge.* Derive the governing equations and the von Karman's-type approximation for boundary layer over a 30^o wedge of a liquid stream of density ρ and viscosity η, approaching at velocity V.

8.9. *Boundary layer along conical body.* Repeat Problem 8.8 for the corresponding axisymmetric case when a liquid stream approaches at velocity V and overtakes a cone of angle ϕ=30^o placed with its leading sharp end facing the stream.

8.8 References

1. H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York, 1950.
2. H. Blasius, "Grenzschlichten in Flussigkeiten mit Kleiner Reibung," *Z. Math. Phys.* **56**, 1 (1908).
3. T. von Karman, "Uber laminare and turbulente reibung," *Z. Angew. Math. Mech.* **1**, 233 (1921).
4. B.C. Sakiades, "Fluid particle mechanics," in *Perry's Chemical Engineers' Handbook*, McGraw-Hill, New York, 1984.
5. V.M. Falkner and S.W. Skan, *Aero. Res. Coun.*, Rep. and Mem. **1314**, 1 (1930).
6. G.K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, 1979.
7. K. Hiemenz, *Die grenzschicht an einem in den gleichformigen flüssigkeitsstrom eingetauchten geraden krieszylinder*, Ph.D. Thesis, University of Göttingen, 1911.
8. F. Homann, "Der einfluss grösser zähigkeit bei der strömung um den Zylinder und um die kugel," *Z. Angew. Math. Mech.* **16**, 153 (1936).
9. L. Howarth, "On the calculation of steady flow in the boundary layer near the surface of a cylinder in a stream," *Aeron. Res. Council-Britain*, 1962.
10. W.G. Cochran, "The flow due to a rotating disc," *Proc. Cambr. Phil. Soc.* **30**, 365 (1934).