Assignment III

Fall 2004

<u>INSTRUCTIONS</u>:

- There are four problems in this assignment.
- You will be graded on how you arrived at the answers. **Show all your work**.
- Due Time: 4:30 p.m. on Thursday, November 18, 2004. A hard copy of the report is encouraged.

1. (20%) Consider the linear system

$$\begin{aligned} 10^{-4}x_1 + x_2 &= b_1 \\ x_1 + x_2 &= b_2, \end{aligned}$$

where $b_1,\ b_2 \neq 0$.

(a) Find its exact solution.

(b) Let $b_1 = 1$ and $b_2 = 2$. Solve the system using naive Gaussian elimination with three-digit (rounded) arithmetic and compare with the exact solution $x_1 = 1.00010...$ and $x_2 = 0.999899....$

(c) Repeat the preceding part after interchanging the order of the two equations.

(d) Find values of b_1 and b_2 in the original system so that naive Gaussian elimination does not give poor answers.

2. (20%) Consider the linear system of equations

$$\begin{array}{rrrrcr} -x_1 & +x_2 & & -3x_4 & = & 4 \\ x_1 & & +3x_3 & +x_4 & = & 0 \\ & x_2 & -x_3 & -x_4 & = & 3 \\ 3x_1 & & +x_3 & +2x_4 & = & 1 \end{array}$$

Solve the system using Gaussian elimination with partial pivoting. Show all intermediate steps and write down the index vector at each step. Compare your solution with that obtained via standard MatLab or Maple subroutines. What is the condition number of the coefficient matrix under the Frobenius norm?

3. (20%)

$$Ax = b$$

we consider the iterative scheme

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b, \quad k \geq 1,$$

where the matrix Q is nonsingular.

(a) If $||I - Q^{-1}A|| < 1$ for some subordinate matrix norm, show that the sequence produced by the above scheme converges to the solution of the system for any initial vector $x^{(0)}$.

(b) If $\sigma = ||I - Q^{-1}A|| < 1$, show that

$$||x^{(k)} - x|| \leq \frac{\sigma}{1-\sigma}||x^{(k)} - x^{(k-1)}||.$$

4. (20%) Given singular matrix

$$S = \begin{pmatrix} 10 & -1.6 & 0.6 \\ 0 & 1.2 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Find its singular-value decomposition.

(b) Find its pseudoinverse.

5. (20%) Consider the matrix

$$A = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$$

Factor A in the following ways:

(a) $A = LU$, where L is unit lower triangular and U is upper triangular;

(b) $A = \tilde{L}\tilde{U}$, where $\tilde{L}$ is lower triangular and $\tilde{U}$ is unit upper triangular;

(c) $A = LD\tilde{U}$, where L is unit lower triangular, D is diagonal, and $\tilde{U}$ is unit upper triangular;

(d) $A = \hat{L}(\hat{L})^T$, where $\hat{L}$ is lower triangular;

(e) Solve the system $Ax = b$, where $b = (1, 0, 0, 1)^T$, using one of the above factorizations.