Solve this problem by first showing that there exists a set of appropriate eigenfunctions for this PDE given by

$$\beta_n(x) = \frac{1}{\sqrt{x}} \sin\left(\frac{\pi n \ln x}{\ln 2}\right)$$

where n is an integer. Develop a series solution for the initial boundary value problem using these eigenfunctions.

Consider the solution of the heat equation for the temperature in a rod given by $\phi(x, t)$ but with a variable diffusivity:

$$\phi_t = A^2 \frac{\partial}{\partial x}\left(x^2 \frac{\partial \phi}{\partial x}\right),$$

where A is a constant. Suppose the rod occupies the interval $1 \leq x \leq 2$ and the boundary conditions are given by

$$\phi(1, t) = 0 \qquad \phi(2, t) = 0,$$

and the initial condition is

$$\phi(x, 0) = f(x).$$