Let R be any commutative ring and S a subset of $R \setminus \{0\}$ containing no zero divisors.

Let X be the Cartesian product $R \times S$ and define a relation $\sim$ on X where $(a, b) \sim (c, d)$.

(a) Show that $\sim$ is an equivalence relation on X.

(b) Denote the equivalence class of (a, b) by a/b and the set of equivalence classes by R_S (called the *localization* of R at S). Show that R_S is a commutative ring with 1.

(c) If $a \in S$ show that $\{ra/a\colon r \in R\}$ is a subring of R_S and that $r \mapsto ra/a$ is a monomorphism, so that R can be identified with a subring with R_S.

(d) Show that every $s \in S$ is a unit in R_S.

(e) Give a "universal" definition for the ring R_S and show that R_S is unique up to isomorphism.