6. Convergence. Open and Closed Sets

6.1. Closure of a set. Limit points. By the *open sphere* (or *open ball*) $S(x_0, r)$ in a metric space R we mean the set of points $x \in R$ satisfying the

inequality

$$\rho(x_0, x) < r$$

(ρ is the metric of R).[4] The fixed point x_0 is called the *center* of the sphere, and the number r is called its *radius*. By the *closed sphere* (or *closed ball*) $S[x_0, r]$ with center x_0 and radius r we mean the set of points $x \in R$ satisfying the inequality

$$\rho(x_0, x) \leqslant r.$$

An open sphere of radius ε with center x_0 will also be called an *ε-neighborhood* of x_0, denoted by $O_\varepsilon(x_0)$.

A point $x \in R$ is called a *contact point* of a set $M \subset R$ if every neighborhood of x contains at least one point of M. The set of all contact points of a set M is denoted by $[M]$ and is called the *closure* of M. Obviously $M \subset [M]$, since every point of M is a contact point of M. By the *closure operator* in a metric space R, we mean the mapping of R into R carrying each set $M \subset R$ into its closure $[M]$.

THEOREM 1. *The closure operator has the following properties*:

1) *If* $M \subset N$, *then* $[M] \subset [N]$;
2) $[[M]] = [M]$;
3) $[M \cup N] = [M] \cup [N]$;
4) $[\varnothing] = \varnothing$.

Proof. Property 1) is obvious. To prove property 2), let $x \in [[M]]$. Then any given neighborhood $O_\varepsilon(x)$ contains a point $x_1 \in [M]$. Consider the sphere $O_{\varepsilon_1}(x_1)$ of radius

$$\varepsilon_1 = \varepsilon - \rho(x, x_1).$$

Clearly $O_{\varepsilon_1}(x_1)$ is contained in $O_\varepsilon(x)$. In fact, if $z \in O_{\varepsilon_1}(x_1)$, then $\rho(z, x_1) < \varepsilon_1$ and hence, since $\rho(x, x_1) = \varepsilon - \varepsilon_1$, it follows from the triangle inequality that

$$\rho(z, x) < \varepsilon_1 + (\varepsilon - \varepsilon_1) = \varepsilon,$$

i.e., $z \in O_\varepsilon(x)$. Since $x_1 \in [M]$, there is a point $x_2 \in M$ in $O_{\varepsilon_1}(x)$. But then $x_2 \in O_\varepsilon(x)$ and hence $x \in [M]$, since $O_\varepsilon(x)$ is an arbitrary neighborhood of x. Therefore $[[M]] \subset [M]$. But obviously $[M] \subset [[M]]$ and hence $[[M]] = [M]$, as required.

To prove property 3), let $x \in [M \cup N]$ and suppose $x \notin [M] \cup [N]$. Then $x \notin [M]$ and $x \notin [N]$. But then there exist neighborhoods $O_{\varepsilon_1}(x)$ and $O_{\varepsilon_2}(x)$ such that $O_{\varepsilon_1}(x)$ contains no points of M while $O_{\varepsilon_2}(x)$ contains

[4] Any confusion between "sphere" meant in the sense of spherical surface and "sphere" meant in the sense of a solid sphere (or ball) will always be avoided by judicious use of the adjectives "open" or "closed."

no points of N. It follows that the neighborhood $O_\varepsilon(x)$, where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, contains no points of either M or N, and hence no points of $M \cup N$, contrary to the assumption that $x \in [M \cup N]$. Therefore $x \in [M] \cup [N]$, and hence

$$[M \cup N] \subset [M] \cup [N], \tag{1}$$

since x is an arbitrary point of $[M \cup N]$. On the other hand, since $M \subset M \cup N$ and $N \subset M \cup N$, it follows from property 1) that $[M] \subset [M \cup N]$ and $[N] \subset [M \cup N]$. But then

$$[M] \cup [N] \subset [M \cup N],$$

which together with (1) implies $[M \cup N] = [M] \cup [N]$.

Finally, to prove property 4), we observe that given any $M \subset R$,

$$[M] = [M \cup \varnothing] = [M] \cup [\varnothing],$$

by property 3). It follows that $[\varnothing] \subset [M]$. But this is possible for arbitrary M only if $[\varnothing] = \varnothing$. (Alternatively, the set with no elements can have no contact points!) ▮

A point $x \in R$ is called a *limit point* of a set $M \subset R$ if every neighborhood of x contains infinitely many points of M. The limit point may or may not belong to M. For example, if M is the set of rational numbers in the interval $[0, 1]$, then every point of $[0, 1]$, rational or not, is a limit point of M.

A point x *belonging* to a set M is called an *isolated point* of M if there is a ("sufficiently small") neighborhood of x containing no points of M other than x itself.

6.2. Convergence and limits. A sequence of points $\{x_n\} = x_1, x_2, \ldots, x_n, \ldots$ in a metric space R is said to *converge* to a point $x \in R$ if every neighborhood $O_\varepsilon(x)$ of x contains all points x_n starting from a certain index (more exactly, if, given any $\varepsilon > 0$, there is an integer N_ε such that $O_\varepsilon(x)$ contains all points x_n with $n > N_\varepsilon$). The point x is called the *limit* of the sequence $\{x_n\}$, and we write $x_n \to x$ (as $n \to \infty$). Clearly, $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} \rho(x, x_n) = 0.$$

It is an immediate consequence of the definition of a limit that

1) No sequence can have two distinct limits;
2) If a sequence $\{x_n\}$ converges to a point x, then so does every subsequence of $\{x_n\}$

(give the details).

THEOREM 2. *A necessary and sufficient condition for a point x to be a contact point of a set M is that there exist a sequence $\{x_n\}$ of points of M converging to x.*

Proof. The condition is necessary, since if x is a contact point of M, then every neighborhood $O_{1/n}(x)$ contains at least one point $x_n \in M$, and these points form a sequence $\{x_n\}$ converging to M. The sufficiency is obvious. ∎

THEOREM 2′. *A necessary and sufficient condition for a point x to be a limit point of a set M is that there exist a sequence $\{x_n\}$ of distinct points of M converging to x.*

Proof. Clearly, if x is a limit point of M, then the points $x_n \in O_{1/n}(x) \cap M$ figuring in the proof of Theorem 2 can be chosen to be distinct. This proves the necessity, and the sufficiency is again obvious. ∎

6.3. Dense subsets. Separable spaces. Let A and B be two subsets of a metric space R. Then A is said to be *dense* in B if $[A] \supset B$. In particular, A is said to be *everywhere dense* (in R) if $[A] = R$. A set A is said to be *nowhere dense* if it is dense in no (open) sphere at all.

Example 1. The set of all rational points is dense in the real line R^1.

Example 2. The set of all points $x = (x_1, x_2, \ldots, x_n)$ with rational coordinates is dense in each of the spaces R^n, R_0^n and R_1^n introduced in Examples 3–5, pp. 38–39.

Example 3. The set of all points $x = (x_1, x_2, \ldots, x_k, \ldots)$ with only finitely many nonzero coordinates, each a rational number, is dense in the space l_2 introduced in Example 7, p. 39.

Example 4. The set of all polynomials with rational coefficients is dense in both spaces $C_{[a,b]}$ and $C_{[a,b]}^2$ introduced in Examples 6 and 8, pp. 39 and 40.

DEFINITION. *A metric space is said to be* ***separable*** *if it has a countable everywhere dense subset.*

Example 5. The spaces R^1, R^n, R_0^n, R_1^n, l_2, $C_{[a,b]}$, and $C_{[a,b]}^2$ are all separable, since the sets in Examples 1–4 above are all countable.

Example 6. The "discrete space" M described in Example 1, p. 38 contains a countable everywhere dense subset and hence is separable if and only if it is itself a countable set, since clearly $[M] = M$ in this case.

Example 7. There is no countable everywhere dense set in the space m of all bounded sequences, introduced in Example 9, p. 41. In fact, consider the set E of all sequences consisting exclusively of zeros and ones. Clearly, E has the power of the continuum (recall Theorem 6, Sec. 2.5), since there is a one-to-one correspondence between E and the set of all subsets of the set $Z_+ = \{1, 2, \ldots, n, \ldots\}$ (describe the correspondence). According to formula (12), p. 41, the distance between any two points of E equals 1. Suppose we surround each point of E by an open sphere of radius $\frac{1}{2}$, thereby obtaining an uncountably infinite family of pairwise disjoint spheres. Then if some set M is everywhere dense in m, there must be at least one point of M in each of the spheres. It follows that M cannot be countable and hence that m cannot be separable.

6.4. Closed sets. We say that a subset M of a metric space R is *closed* if it coincides with its own closure, i.e., if $[M] = M$. In other words, a set is called closed if it contains all its limit points (see Problem 2).

Example 1. The empty set $\varnothing$ and the whole space R are closed sets.

Example 2. Every closed interval $[a, b]$ on the real line is a closed set.

Example 3. Every closed sphere in a metric space is a closed set. In particular, the set of all functions f in the space $C_{[a,b]}$ such that $|f(t)| \leqslant K$ (where K is a constant) is closed.

Example 4. The set of all functions f in $C_{[a,b]}$ such that $|f(t)| < K$ (an open sphere) is not closed. The closure of this set is the closed sphere in the preceding example.

Example 5. Any set consisting of a finite number of points is closed.

THEOREM 3. *The intersection of an arbitrary number of closed sets is closed. The union of a* ***finite*** *number of closed sets is closed.*

Proof. Given arbitrary sets F_α indexed by a parameter α, let x be a limit point of the intersection

$$F = \bigcap_\alpha F_\alpha.$$

Then any neighborhood $O_\varepsilon(x)$ contains infinitely many points of F, and hence infinitely many points of each F_α. Therefore x is a limit point of each F_α and hence belongs to each F_α, since the sets F_α are all closed. It follows that $x \in F$, and hence that F itself is closed.

Next let

$$F = \bigcup_{k=1}^{n} F_k$$

be the union of a finite number of closed sets F_k, and suppose x does not belong to F. Then x does not belong to any of the sets F_k, and hence cannot be a limit point of any of them. But then, for every k, there is a neighborhood $O_{\varepsilon_k}(x)$ containing no more than a finite number of points of F_k. Choosing

$$\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\},$$

we get a neighborhood $O_\varepsilon(x)$ containing no more than a finite number of points of F, so that x cannot be a limit point of F. This proves that a point $x \notin F$ cannot be a limit point of F. Therefore F is closed. ▮

6.5. Open sets. A point x is called an *interior point* of a set M if x has a neighborhood $O_\varepsilon(x) \subset M$, i.e., a neighborhood consisting entirely of points of M. A set is said to be *open* if its points are all interior points.

Example 1. Every open interval (a, b) on the real line is an open set. In fact, if $a < x < b$, choose $\varepsilon = \min\{x - a, b - x\}$. Then clearly $O_\varepsilon(x) \subset (a, b)$.

Example 2. Every open sphere $S(a, r)$ in a metric space is an open set. In fact, $x \in S(a, r)$ implies $\rho(a, x) < r$. Hence, choosing $\varepsilon = r - \rho(a, x)$, we have $O_\varepsilon(x) = S(x, \varepsilon) \subset S(a, r)$.

Example 3. Let M be the set of all functions f in $C_{[a,b]}$ such that $f(t) < g(t)$, where g is a fixed function in $C_{[a,b]}$. Then M is an open subset of $C_{[a,b]}$.

THEOREM 4. *A subset M of a metric space R is open if and only if its complement $R - M$ is closed.*

Proof. If M is open, then every point $x \in M$ has a neighborhood (entirely) contained in M. Therefore no point $x \in M$ can be a contact point of $R - M$. In other words, if x is a contact point of $R - M$, then $x \in R - M$, i.e., $R - M$ is closed.

Conversely, if $R - M$ is closed, then any point $x \in M$ must have a neighborhood contained in M, since otherwise every neighborhood of x would contain points of $R - M$, i.e., x would be a contact point of $R - M$ not in $R - M$. Therefore M is open. ▮

COROLLARY. *The empty set $\varnothing$ and the whole space R are open sets.*

Proof. An immediate consequence of Theorem 4 and Example 1, Sec. 6.4. ▮

THEOREM 5. *The union of an arbitrary number of open sets is open. The intersection of a finite number of open sets is open.*

Proof. This is the "dual" of Theorem 3. The proof is an immediate consequence of Theorem 4 and formulas (3)–(4), p. 4. ▮

6.6. Open and closed sets on the real line. The structure of open and closed sets in a given metric space can be quite complicated. This is true even for open and closed sets in a Euclidean space of two or more dimensions (R^n, $n \geqslant 2$). In the one-dimensional case, however, it is an easy matter to give a complete description of all open sets (and hence of all closed sets):

THEOREM 6. *Every open set G on the real line is the union of a finite or countable system of pairwise disjoint open intervals.*[5]

Proof. Let x be an arbitrary point of G. By the definition of an open set, there is at least one open interval containing x and contained in G. Let I_x be the union of all such open intervals. Then, as we now show, I_x is itself an open interval. In fact, let[6]

$$a = \inf I_x, \qquad b = \sup I_x$$

(where we allow the cases $a = -\infty$ and $b = +\infty$). Then obviously

$$I_x \subset (a, b). \tag{2}$$

Moreover, suppose y is an arbitrary point of (a, b) distinct from x, where, to be explicit, we assume that $a < y < x$. Then there is a point $y' \in I_x$ such that $a < y' < y$ (why?). Hence G contains an open interval containing the points y' and x. But then this interval also contains y, i.e., $y \in I_x$. (The case $y > x$ is treated similarly.) Moreover, the point x belongs to I_x, by hypothesis. It follows that $I_x \supset (a, b)$, and hence by (2) that $I_x = (a, b)$. Thus I_x is itself an open interval, as asserted, in fact the open interval (a, b).

By its very construction, the interval (a, b) is contained in G and is not a subset of a larger interval contained in G. Moreover, it is clear that two intervals I_x and $I_{x'}$ corresponding to distinct points x and x' either coincide or else are disjoint (otherwise I_x and $I_{x'}$ would both be contained in a larger interval $I_x \cup I_{x'} = I \subset G$. There are no more than countably many such pairwise disjoint intervals I_x. In fact, choosing an arbitrary rational point in each I_x, we establish a one-to-one correspondence between the intervals I_x and a subset of the rational numbers. Finally, it is obvious that

$$G = \bigcup_x I_x. \quad \blacksquare$$

COROLLARY. *Every closed set on the real line can be obtained by deleting a finite or countable system of pairwise disjoint intervals from the line.*

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[5] The infinite intervals $(-\infty, \infty)$, (a, ∞), and $(-\infty, b)$ are regarded as open.

[6] Given a set of real numbers E, $\inf E$ denotes the greatest lower bound or infimum of E, while $\sup E$ denotes the least upper bound or supremum of E.

Proof. An immediate consequence of Theorems 4 and 6. ▮

Example 1. Every closed interval $[a, b]$ is a closed set (here a and b are necessarily finite).

Example 2. Every single-element set $\{x_0\}$ is closed.

Example 3. The union of a finite number of closed intervals and single-element sets is a closed set.

Example 4 (The Cantor set). A more interesting example of a closed set on the line can be constructed as follows: Delete the open interval $(\frac{1}{3}, \frac{2}{3})$ from the closed interval $F_0 = [0, 1]$, and let F_1 denote the remaining closed set, consisting of two closed intervals. Then delete the open intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ from F_1, and let F_2 denote the remaining closed set, consisting of four closed intervals. Then delete the "middle third" from each of these four intervals, getting a new closed set F_3, and so on (see Figure 9). Continuing this process indefinitely, we get a sequence of closed sets F_n such that

$$F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$$

(such a sequence is said to be *decreasing*). The intersection

$$F = \bigcap_{n=0}^{\infty} F_n$$

of all these sets is called the *Cantor set*. Clearly F is closed, by Theorem 3, and is obtained from the unit interval $[0, 1]$ by deleting a countable number of open intervals. In fact, at the nth stage of the construction, we delete 2^{n-1} intervals, each of length $1/3^n$.

To describe the structure of the set F, we first note that F contains the points

$$0, 1, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{9}, \tfrac{2}{9}, \tfrac{7}{9}, \tfrac{8}{9}, \ldots, \tag{3}$$

i.e., the end points of the deleted intervals (together with the points 0 and 1).

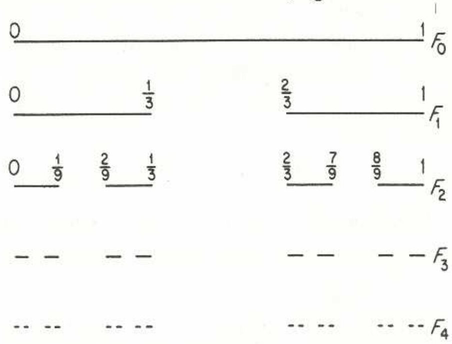

FIGURE 9

However F contains many other points. In fact, given any $x \in [0, 1]$, suppose we write x in ternary notation, representing x as a series

$$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_n}{3^n} + \cdots, \tag{4}$$

where each of the numbers $a_1, a_2, \ldots, a_n, \ldots$ can only take one of the three values 0, 1, 2. Then it is easy to see that x belongs to F if and only if x has a representation (4) such that none of the numbers $a_1, a_2, \ldots, a_n, \ldots$ equals 1 (think things through).[7]

Remarkably enough, the set F has the power of the continuum, i.e., there are as many points in F as in the whole interval $[0, 1]$, despite the fact that the sum of the lengths of the deleted intervals equals

$$\tfrac{1}{3} + \tfrac{2}{9} + \tfrac{4}{27} + \cdots = 1.$$

To see this, we associate a new point

$$y = \frac{b_1}{2} + \frac{b_2}{2^2} + \cdots + \frac{b_n}{2^n} + \cdots$$

with each point (4), where[8]

$$b_n = \begin{cases} 0 & \text{if} \quad a_n = 0, \\ 1 & \text{if} \quad a_n = 2. \end{cases}$$

In this way, we set up a one-to-one correspondence between F and the whole interval $[0, 1]$. It follows that F has the power of the continuum, as asserted. Let A_1 be the set of points (3). Then $F = A_1 \cup A_2$, where the set $A_2 = F - A_1$ is uncountable, since A_1 is countable and F itself is not. The points of A_1 are often called "points (of F) of the first kind," while those of A_2 are called "points of the second kind."

Problem 1. Give an example of a metric space R and two open spheres $S(x, r_1)$ and $S(y, r_2)$ in R such that $S(x, r_1) \subset S(y, r_2)$ although $r_1 > r_2$.

Problem 2. Prove that every contact point of a set M is either a limit point of M or an isolated point of M.

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[7] Just as in the case of ordinary decimals, certain numbers can be written in two distinct ways. For example,

$$\frac{1}{3} = \frac{1}{3} + \frac{0}{3^2} + \frac{0}{3^3} + \cdots + \frac{0}{3^n} + \cdots = \frac{0}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots + \frac{2}{3^n} + \cdots.$$

Since none of the numerators in the second representation equals 1 the point $\frac{1}{3}$ belongs to F (this is already obvious from the construction of F).

[8] If x has two representations of the form (4), then one and only one of them has no numerators $a_1, a_2, \ldots, a_n, \ldots$ equal to 1. These are the numbers used to define b_n.

Comment. In particular, $[M]$ can only contain points of the following three types:

a) Limit points of M belonging to M;
b) Limit points of M which do not belong to M;
c) Isolated points of M.

Thus $[M]$ is the union of M and the set of all its limit points.

Problem 3. Prove that if $x_n \to x$, $y_n \to y$ as $n \to \infty$, then $\rho(x_n, y_n) \to \rho(x, y)$.

Hint. Use Problem 1a, p. 45.

Problem 4. Let f be a mapping of one metric space X into another metric space Y. Prove that f is continuous at a point x_0 if and only if the sequence $\{y_n\} = \{f(x_n)\}$ converges to $y = f(x_0)$ whenever the sequence $\{x_n\}$ converges to x_0.

Problem 5. Prove that

a) The closure of any set M is a closed set;
b) $[M]$ is the smallest closed set containing M.

Problem 6. Is the union of infinitely many closed sets necessarily closed? How about the intersection of infinitely many open sets? Give examples.

Problem 7. Prove directly that the point $\frac{1}{4}$ belongs to the Cantor set F, although it is not an end point of any of the open intervals deleted in constructing F.

Hint. The point $\frac{1}{4}$ divides the interval $[0, 1]$ in the ratio 1:3. It also divides the interval $[0, \frac{1}{3}]$ left after deleting $(\frac{1}{3}, \frac{2}{3})$ in the ratio 3:1, and so on.

Problem 8. Let F be the Cantor set. Prove that

a) The points of the first kind, i.e., the points (3) form an everywhere dense subset of F;
b) The numbers of the form $t_1 + t_2$, where $t_1, t_2 \in F$, fill the whole interval $[0, 2]$.

Problem 9. Given a metric space R, let A be a subset of R and x a point of R. Then the number

$$\rho(A, x) = \inf_{a \in A} \rho(a, x)$$

is called the *distance between A and x*. Prove that

a) $x \in A$ implies $\rho(A, x) = 0$, but not conversely;
b) $\rho(A, x)$ is a continuous function of x (for fixed A);
c) $\rho(A, x) = 0$ if and only if x is a contact point of A;
d) $[A] = A \cup M$, where M is the set of all points x such that $\rho(A, x) = 0$.

Problem 10. Let A and B be two subsets of a metric space R. Then the number

$$\rho(A, B) = \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$$

is called the *distance between A and B*. Show that $\rho(A, B) = 0$ if $A \cap B \neq \varnothing$, but not conversely.

Problem 11. Let M_K be the set of all functions f in $C_{[a,b]}$ satisfying a *Lipschitz condition*, i.e., the set of all f such that

$$|f(t_1) - f(t_2)| \leqslant K|t_1 - t_2|$$

for all $t_1, t_2 \in [a, b]$, where K is a fixed positive number. Prove that

a) M_K is closed and in fact is the closure of the set of all differentiable functions on $[a, b]$ such that $|f'(t)| \leqslant K$;
b) The set
$$M = \bigcup_K M_K$$
of all functions satisfying a Lipschitz condition for some K is not closed;
c) The closure of M is the whole space $C_{[a,b]}$.

Problem 12. An open set G in n-dimensional Euclidean space R^n is said to be *connected* if any points $x, y \in G$ can be joined by a polygonal line[9] lying entirely in G. For example, the (open) disk $x^2 + y^2 < 1$ is connected, but not the union of the two disks

$$x^2 + y^2 < 1, \qquad (x - 2)^2 + y^2 < 1$$

(even though they share a contact point). An open subset of an open set G is called a *component* of G if it is connected and is not contained in a larger connected subset of G. Use Zorn's lemma to prove that every open set G in R^n is the union of no more than countably many pairwise disjoint components.

Comment. In the case $n = 1$ (i.e., on the real line) every connected open set is an open interval, possibility one of the infinite intervals $(-\infty, \infty)$, (a, ∞), $(-\infty, b)$. Thus Theorem 6 on the structure of open sets on the line is tantamount to two assertions:

1) Every open set on the line is the union of a finite or countable number of components;
2) Every open connected set on the line is an open interval.

[9] By a *polygonal line* we mean a curve obtained by joining a finite number of straight line segments end to end.

The first assertion holds for open sets in R^n (and in fact is susceptible to further generalizations), while the second assertion pertains specifically to the real line.

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