Def 7.13 A stochastic process is a collection $\{X_t, t \in T\}$ of random variables on a prob. space $(\Omega, \mathcal{U}, \mathbb{P})$

Def 7.14 $\sigma(X)$: σ-algebra generated by the events

$$(X_{t_1}, \dots, X_{t_k}) \in A \qquad A \in \mathcal{B}(\mathbb{R}^k)$$

Thm 7.15 The (probability) law of X is uniquely determined by its finite dimensional distributions

$$\mu_{t_1, \dots, t_k}(A) = \mathbb{P}[(X_{t_1}, \dots, X_{t_k}) \in A]$$

$$A \in \mathcal{B}(\mathbb{R}^k)$$

Def 7.16 Let $(X_t)_{t \in T}$ be a Gaussian Process

The covariance function of X is the function

$$\Gamma: T \times T \longrightarrow \mathbb{R} \qquad \text{defined by}$$

$$\Gamma(s,t) = \mathrm{Cov}[X_s, X_t] = \mathbb{E}[X_s X_t]$$

Thm 7.17

The law of X is uniquely determined by the function Γ

Def 7.18

We say that a function Γ on $T \times T$ is symmetric and of positive type iif

- $\Gamma(s,t) = \Gamma(t,s)$

and

- if c is a function with finite support on T

$$\sum_{T \times T} c(s)\, c(t)\, \Gamma(s,t) \geqslant 0$$

$\left(\overset{''}{\mathbb{E}}\left[\left(\sum_T c(s) X(s) \right)^2 \right] \text{ if } \Gamma \text{ is the cov. function of } X \right)$

Thm 7.19

Let Γ be a symmetric function of positive type on $T \times T$, then there exists a Gaussian process $(X_t)_{t\in T}$ whose covariance ~~matrix~~ function is Γ

proof: Kolmogorov existence theorem

Rk $T = \mathbb{R}_+$, $\rightarrow$ $(X_t)_{t\in T}$ is a Brownian motion
$\Gamma(s,t) = s \wedge t$

$T = [0,1]^2 \rightarrow$ Brownian sheet

$T = \mathbb{R}^d \rightarrow$ Gaussian field

$T = \mathbb{L}^2(E, \mathcal{E}, \mu)$

$\Gamma(f, g) = \langle f, g \rangle_{\mathbb{L}^2}$ $\longmapsto$ Gaussian measure

$= \int_E f(x) g(x) \mu(dx)$

Def 7.20

For $T = \mathbb{R}_+$ & $\Gamma(s,t) = s \wedge t$ $(\min(s,t))$

The process $(X_t, t \in \mathbb{R}_+)$ noted $(B_t, t \in \mathbb{R}_+)$ is called a Brownian Motion

Proposition 7.21

A process $(X_t, t \in \mathbb{R}_+)$ is a Brownian Motion iif

(i) $X_0 = 0$ a.s.

(ii) $\forall\, 0 \leqslant s \leqslant t$ the random variable $X_t - X_s$ is independent from $\sigma(X_r, r \leqslant s)$ and has a $\mathcal{N}(0, t-s)$ prob. dist.

Corollary 7.22

The process $X = (X_t, t \in \mathbb{R}_+)$ is a B.M. (Brownian Motion)

iif $X_0 = 0$ a.s. and

$\forall\ 0 = t_0 < t_1 < \cdots < t_n$ the random variables

$X_{t_i} - X_{t_{i-1}}$ $(i \in \{1, \dots, n\})$ are independent

$X_{t_i} - X_{t_{i-1}}$ is $\mathcal{N}(0, t_i - t_{i-1})$

In particular the (prob.) density of $(X_{t_1}, \dots, X_{t_n})$ is

$$f(y_1, \dots, y_n) = \frac{1}{(2\pi)^{n/2} \left(t_1 (t_2 - t_1) \dots (t_n - t_{n-1})\right)^{1/2}} \exp\left[- \sum_{i=1}^{n} \frac{(y_i - y_{i-1})^2}{2(t_i - t_{i-1})}\right]$$

Prop 1.23

Let B be a B.M., then

(i) $-B$ is a B.M.

(ii) $\forall \lambda > 0$, the process

$$B_t^\lambda = \frac{1}{\lambda} B_{\lambda^2 t}$$ is also a B.M.

(iii) $\forall s \geq 0$ the process $B_t^{(s)} = B_{t+s} - B_s$ is a B.M. indep. from $\sigma(B_r, r \leq s)$

Ex:

Let $(Y_i)_{i\in\mathbb{N}}$ be an infinite sequence of iid rand. var such that $\mathbb{E}[Y_i]=0$ and $\mathbb{E}[Y_i^2]=1$

Let $t\in\mathbb{R}^+$, write

$$S_n := \sum_{i=1}^{n} Y_i$$

a) Is $\frac{S_n}{n}$ converging as $n\to\infty$?

If yes specify the limit and the type of convergence (a.s? in prob.? in law?)

b) For $t\in T:=[0,1]$ write $[nt]$ the integer part of nt and

$$X_n(t) := \frac{S_{[nt]}}{\sqrt{n}}$$

Is $X_n(t)$ converging as $n\to\infty$? If yes explain how, why and specify the limit.

c) Let $0=t_0<t_1<\dots<t_m=1$

Is $(X_n(t_0),\dots,X_n(t_m))$ converging in law as $n\to\infty$?

If yes explain why and specify the limit law

d) It is known (admit this point) that the stochastic process $(X_n(t))_{t\in[0,1]}$ converges in law towards a stochastic process $(Z(t))_{t\in[0,1]}$ iif

- $\forall \varepsilon>0,\ \forall \eta>0\ \ \exists \delta>0 /$

$$\limsup_{n\to\infty} \mathbb{P}[\omega(X_n,\delta)>\eta] \leqslant \varepsilon$$

(⇒ uniform tightness of $((\text{law}(X_n))_{n\in\mathbb{N}})$

where $\omega(X_n,\delta) := \sup_{|t-s|\leqslant\delta} |X_n(t)-X_n(s)|$

is the modulus of continuity of X_n

- And for all m & $0=t_0<t_1<\cdots<t_m=1$ the law of $(X_n(t_0),\dots,X_n(t_m))$ converges towards the law of $(Z(t_0),\dots,Z(t_m))$

Does $(X_n(t))_{t\in[0,1]} \xrightarrow[n\to\infty]{\text{law}} (Z(t))_{t\in[0,1]}$?

If yes what is Z ?

7.3 Gaussian Measure

Motivation: What is / How to define $\int_0^t f(s)\, dB_s$

dB_s ↓ BM indexed by $\mathbb{R}, \mathbb{R}^2, \dots, \mathbb{R}^n$

Let $(E, \mathcal{E})$ be a measurable space and μ a $\underbrace{\sigma\text{-finite}}_{\left(\exists (A_i)_{1 \le i \le \infty} \,/\, E = \bigcup_{i=1}^{\infty} A_i \;\&\; \forall i\; \mu(A_i) < \infty\right)}$ measure on $(E, \mathcal{E})$

Def 7.24

A Gaussian measure of intensity μ is an isometry G ((linear mapping preserving the inner product) from $\mathbb{L}^2(E, \mathcal{E}, \mu)$ to a Gaussian space.

$$G : \mathbb{L}^2(E, \mathcal{E}, \mu) \longrightarrow H$$

$\forall \lambda \in \mathbb{R},\ f, g \in \mathbb{L}^2(E, \mathcal{E}, \mu)$

$$G(\lambda f + g) = \lambda\, G(f) + G(g)$$

$$\mathbb{E}[G(f)\, G(g)] = \langle f, g \rangle_{\mathbb{L}^2(E, \mathcal{E}, \mu)}$$

$\Rightarrow$ If $f \in L^2(E, \mathcal{E}, \mu)$

$G(f)$ is a centered Gaussian random variable of variance

$$\mathbb{E}[G(f)^2] = \| G(f) \|^2_{\mathbb{L}^2(\Omega)} = \|f\|^2_{\mathbb{L}^2(E,\mu)}$$

In particular if $A \in \mathcal{E}$ with $\mu(A) < \infty$, $f = \mathbb{1}_A$

then $G(\mathbb{1}_A) \underset{\text{notation}}{=} G(A)$ is $\mathcal{N}(0, \mu(A))$

$\rightarrow$ If $A_1, \dots, A_n \in \mathcal{E} \,/\, \mu(A_i) < \infty$ $\quad A_i \cap A_j = \emptyset \quad i \neq j$

then the vector $(G(A_1), \dots, G(A_n))$ is a Gauss. vector in $\mathbb{R}^n$ with diagonal covariance matrix

$i \neq j \quad \mathbb{E}[G(A_i)\, G(A_j)] = \langle \mathbb{1}_{A_i}, \mathbb{1}_{A_j} \rangle_{\mathbb{L}^2(\mu)} = 0$

$\rightarrow$ $G(A_i)$ are indep.

$\rightarrow$ $A = \bigcup_{i=1}^{\infty} A_i \quad A_i \cap A_j = \emptyset \quad i \neq j \quad \mu(A) < \infty$

$$\sum_{j=1}^{n} G(A_j) \xrightarrow[n \to \infty]{\mathbb{L}^2(\Omega)} G(A)$$

isometry

$\rightarrow$ The properties of the mapping $A \to G(A)$ look like the properties of a measure

$\rightarrow$ in general for ω fixed, $A \longrightarrow G(A)(\omega)$ doesn't define a measure

Thm 7.25

If $(E, \mathcal{E})$ is a measurable space, μ a σ-finite measure on $(E, \mathcal{E})$ then there exists a Gaussian measure of intensity μ on $(E, \mathcal{E})$.

proof

If $\mathbb{L}^2(E, \mathcal{E}, \mu)$ is separable (contains a dense countable subset)

Assume

(ex $E = \mathbb{R}_+$ $\mu = dx$)

There exists

φ_n: an orthonormal basis of $\mathbb{L}^2(E, \mathcal{E}, \mu)$

$\varphi_n \in \mathbb{L}^2(E, \mathcal{E}, \mu)$, $\quad \langle \varphi_n, \varphi_m \rangle_{\mathbb{L}^2} = \delta_{nm}$

$\forall f \in \mathbb{L}^2(E, \mathcal{E}, \mu)$

$$f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle_{\mathbb{L}^2} \varphi_n$$

ξ_n : iid $\mathcal{N}(0,1)$

$$G(f) := \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \xi_n$$

→

$G(f)$ is Gaussian

$\forall f, g \in \mathbb{L}^2(E, \mathcal{E}, \mu)$

$$\mathbb{E}[G(f)\, G(g)] = \sum_{m,n=1}^{\infty} \langle f, \varphi_n \rangle \langle f, \varphi_m \rangle \, \delta_{nm}$$

$$= \langle f, g \rangle_{\mathbb{L}^2(E, \mathcal{E}, \mu)}$$

<u>From now on</u> we will consider

$$(E, \mathcal{E}, \mu) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx)$$

$(\varphi_n)_{n \in \mathbb{N}^*}$: orthonormal basis of $\mathbb{L}^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx)$

(Fourier, Haar, Wavelet)

$$G : \mathbb{L}^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx) \longrightarrow H$$

$$f \longrightarrow G(f) = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \xi_n$$

$$\| \qquad\qquad\qquad \downarrow$$

$$\sum_n \langle f, \varphi_n \rangle \varphi_n \qquad \text{by isometry}$$

convergence in $\mathbb{L}^2(\Omega)$

Thm 7.26

Let G be a Gaussian measure on $\mathbb{L}^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dx)$ of intensity the Lebesgue measure.

The process $(B_t, t \in \mathbb{R}_+)$ defined by $B_t = G[\mathbb{1}_{[0,t]}]$ is a Brownian Motion (started from 0, real, 1d)

proof:

B_t is a Gaussian process since $\forall \lambda_1, \dots, \lambda_n$ $\forall n$

$\sum_i \lambda_i B_{t_i} = G[\sum_i \lambda_i \mathbb{1}_{[0,t_i]}]$ is Gaussian

$\Gamma(s,t) = \mathbb{E}[B_s B_t] = \int_{\mathbb{R}_+} \mathbb{1}_{[0,s]} \mathbb{1}_{[0,t]} \, dx = s \wedge t$

$= \mathbb{E}[G[0,s] \, G[0,t]]$

G is an isometry

Construction of the B.M. on $[0,1]$

$(\varphi_n)_{n\geq 1}$ orthonormal basis of $\mathbb{L}^2([0,1], \mathcal{B}([0,1]), dx)$

$$B_t = \sum_{n=1}^{\infty} \int_0^t \varphi_n(s)\,ds \;\; \xi_n \qquad \xi_n \text{: iid } \mathcal{N}(0,1)$$

$$\sum_{n=1}^{M} \underbrace{\int_0^t \varphi_n(s)\,ds}_{\langle \mathbb{1}_{[0,t]}, \varphi_n \rangle_{\mathbb{L}^2([0,1])}} \;\xi_n \xrightarrow[M\to\infty]{\mathbb{L}^2(\Omega)} B_t$$

Examples of orthonormal bases φ_n:

→ Trigonometric (Fourier series)

$$\left\{ 1,\ \sqrt{2}\,\sin(2\pi n t),\ \sqrt{2}\,\cos(2\pi n t),\ n=1,2,\dots \right\}$$

Haar functions

$$\left\{ \varphi_0,\ \varphi_{j,n},\ j = 1,\dots,2^{n-1},\ n=1,2,\dots \right\}$$

$\varphi_0(t) \equiv 1$ and with $k = 2j-1$

$$\varphi_{j,n} = \begin{cases} 2^{\frac{n-1}{2}} & \frac{k-1}{2^n} \leq t < \frac{k}{2^n} \\ -2^{\frac{n-1}{2}} & \frac{k}{2^n} < t \leq \frac{k+1}{2^n} \\ 0 & \text{elsewhere} \end{cases}$$

Def 7.27

If B is a B.M. and G the associated measure. We write for $f \in \mathbb{L}^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$

$$\int_0^\infty f(s)\, dB_s := G(f)$$

$$\int_0^t f(s)\, dB_s := G[f\, \mathbb{1}_{[0,t]}]$$

The mapping $f \longrightarrow \int_0^\infty f(s)\, dB_s$ is called Wiener integral with respect to the B.M. B

Justification of this notation:

if $u < v$

$$\int_u^v dB_s = G(]u, v]) = B_v - B_u$$

Def A function $f \in L^2(0,T)$ is called a step function if there exists a partition $P = \{0 = t_0 < t_1 < \cdots < t_m = T\}$ such that

$f(t) = f_k$ for $t_k \leqslant t < t_{k+1}$ $(k = 0, \ldots, m-1)$

Proposition 7.28

If $f \in L^2(0,T)$ is a step function as above

Then

$$\int_0^T f(s)\, dB_s = \sum_{k=0}^{m-1} f_k \left(B_{t_{k+1}} - B_{t_k} \right)$$

Lemma 7.29

If $f \in L^2(0,T)$, then there exists a sequence of step functions $f_n \in L^2(0,T)$ such that

$$\int_0^T |f - f_n|^2(s)\, ds \xrightarrow[n \to \infty]{} 0$$

Moreover by isometry

$$\int_0^T f_n\, dB_s \xrightarrow[n \to \infty]{\mathbb{L}^2(\Omega)} \int_0^T f(s)\, dB_s$$

Lemma 7.30

$\forall a, b \in \mathbb{R},\ f, g \in L^2(0,T)$

$$\int_0^T (af + bg)\, dB_s = a\int_0^T f(s)\, dB_s + b\int_0^T g(s)\, dB_s$$

$$\mathbb{E}\left[\int_0^T f(s)\, dB_s\right] = 0$$

$$\mathbb{E}\left[\int_0^T f(s)\, dB_s \int_0^T g\, dB_s\right] = \int_0^T f(s)\, g(s)\, ds$$

$\int_0^T f(s)\, dB_s$ is $\mathcal{N}\left(0, \int_0^T f^2(s)\, ds\right)$

7.4 Stochastic Integrals

How to define $\int_0^T X(s,\omega)\, dB_s(\omega)$?

$X(s,\omega)$: stochastic process

Def 7.3.1

A real valued stochastic process X is called progressively measurable with respect to $\mathcal{F}_t = \sigma(B_s, s \leq t)$ if for each time t, $X(t,\omega)$ is $\mathcal{F}_t$-measurable and jointly measurable in the variables t and ω togeather.

Def 7.3.2

We denote by $\mathbb{L}^2(0,T)$ the space of all real valued progressively measurable stochastic process such that

$$\mathbb{E}\left[\int_0^T X^2 dt\right] < \infty$$

Def 7.33

A process $X \in \mathbb{L}^2(0,T)$ is called a step process if there exists a partition $P = \{0 = t_0 < t_1 < \cdots < t_m = T\}$ such that

$$X(t,\omega) = X_k(\omega) \quad \text{for } t_k \leqslant t < t_{k+1} \quad (k = 0, \dots, m-1)$$

where each X_k is $\mathcal{F}_{t_k}$-measurable

Def 7.34

Let $X \in \mathbb{L}^2(0,T)$ be a step process as above

Then

$$\int_0^T X \, dB_s = \sum_{k=0}^{m-1} X_k \left(B(t_{k+1}) - B(t_k) \right)$$

is the Itô stochastic integral of X on the interval $[0,T]$

Lemma 7.35

$\forall a, b \in \mathbb{R}, \quad X, Y \in \mathbb{L}^2(0,T)$

$$\int_0^T (a X_s + b Y_s)\, dB_s = a \int_0^T X_s\, dB_s + b \int_0^T Y_s\, dB_s$$

$$\mathbb{E}\left[\int_0^T X_s\, dB_s\right] = 0$$

$$\mathbb{E}\left[\left(\int_0^T X_s\, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^T X_s^2\, ds\right]$$

Lemma 7.36

If $X \in \mathbb{L}^2(0,T)$, there exists a sequence of bounded step processes $X^n \in \mathbb{L}^2(0,T)$ such that

$$\mathbb{E}\left[\int_0^T |X_s - X_s^n|^2\, ds\right] \xrightarrow[n \to \infty]{} 0$$

Def 7.37

If $X \in \mathbb{L}^2(0,T)$ take a step process X^n as above

Then

$$\mathbb{E}\left[\left(\int_0^T (X_s^n - X_s^m)\, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^T (X_s^n - X_s^m)^2\, ds\right] \xrightarrow{n,m \to \infty} 0$$

So the limit

$$\int_0^T X_s\, dB_s := \lim_{n \to \infty} \int_0^T X_s^n\, dB_s$$

exists in $\mathbb{L}^2(\Omega)$ (Cauch sequence, and the space is complete)

and the definition does not depend upon the particular approximating sequence X_s^n

Thm 7.38

$\forall a, b \in \mathbb{R}, \quad X, Y \in \mathbb{L}^2(0,T)$

$$\int_0^T (a X_s + b Y_s)\, dB_s = a \int_0^T X_s\, dB_s + b \int_0^T Y_s\, dB_s$$

$$\mathbb{E}\left[\int_0^T X_s\, dB_s \right] = 0$$

$$\mathbb{E}\left[\left(\int_0^T X_s\, dB_s \right)^2 \right] = \mathbb{E}\left[\int_0^T X_s^2\, ds \right]$$

$$\mathbb{E}\left[\int_0^T X_s\, dB_s \int_0^T Y_s\, dB_s \right] = \mathbb{E}\left[\int_0^T X_s Y_s\, ds \right]$$