7. Gaussian vectors, spaces, measures, processes

7.1 Gaussian Random Variables

$d=1$

X is $\mathcal{N}(0,1) \iff f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

$\downarrow$ density

$\left.\begin{array}{l} \mathbb{E}[e^{zX}] = e^{z^2/2} \quad \forall z \in \mathbb{R} \\ \mathbb{E}[e^{zX}] \text{ is analytic on } \mathbb{C} \end{array}\right| \Rightarrow \mathbb{E}[e^{zX}] = e^{z^2/2} \quad \forall z \in \mathbb{C}$

$z = i\xi, \quad \xi \in \mathbb{R}$

$$\mathbb{E}[e^{i\xi X}] = e^{-\xi^2/2}$$

From the development

$$\mathbb{E}[e^{i\xi X}] = 1 + i\xi\, \mathbb{E}[X] + \dots + \frac{(i\xi)^n}{n!} \mathbb{E}[X^n] + O(|\xi|^{n+1})$$

which is valid if $\forall p > 0, \quad \mathbb{E}[|X|^p] < \infty$

we compute

$$\mathbb{E}[X] = 0 \qquad \mathbb{E}[X^2] = 1 \qquad \mathbb{E}[X^{2n}] = \frac{(2n)!}{2^n n!}$$

Def 7.1

We say that a real rand. var Y has a Gaussian $\mathcal{N}(m, \sigma^2)$ law iif one of the following equivalent propertie ~~is~~ are satisfied

(i) $Y = \sigma X + m$ where X is $\mathcal{N}(0,1)$

(ii) $f_Y(y) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{(y-m)^2}{2\sigma^2}\right]$

(iii) $\mathbb{E}[e^{i\xi Y}] = \exp\left[im\xi - \dfrac{\sigma^2}{2}\xi^2\right]$

we have then $\mathbb{E}[Y] = m$ $\quad Var[\cancel{\sigma^2} Y] = \sigma^2$

Rk Assume that Y is $\mathcal{N}(m, \sigma^2)$ & Y' is $\mathcal{N}(m', \sigma'^2)$ and Y & Y' are indep.

Then $Y + Y'$ is $\mathcal{N}(m+m', \sigma^2 + \sigma'^2)$

Proposition 7.2

Let (X_n) be a sequ. of Gauss. rand. var such that X_n is $\mathcal{N}(m_n, \sigma_n^2)$

Assume that $X_n \xrightarrow[n\to\infty]{\text{law}} X$

Then

1/ X is $\mathcal{N}(m, \sigma^2)$

$$m_n \xrightarrow[n\to\infty]{} m \qquad \sigma^2 = \lim_{n\to\infty} \sigma_n^2$$

3/ If $X_n \xrightarrow[n\to\infty]{\text{prob.}} X$ then

$$\forall p < \infty \quad X_n \xrightarrow[n\to\infty]{L^p(\Omega)} X \quad \left(\Leftrightarrow \mathbb{E}[|X - X_n|^p] \xrightarrow[n\to\infty]{} 0\right)$$

7.2 Gaussian Vectors

E: Euclidean space of dim d
(E is isomorphic to $\mathbb{R}^d$)
$\langle u, v \rangle$: The scalar product of E

Def 7.3

$X : \Omega \longrightarrow E$ is a Gaussian vector iif $\forall u \in E$, $\langle u, X \rangle$ is a Gaussian random variable

Ex: If $E = \mathbb{R}^d$ & $X_1, \dots, X_d$ are indep Gauss. rand var. then $X = (X_1, \dots, X_d)$ is a Gauss. vector

Def 7.4

A mapping $B : E \times E \longrightarrow \mathbb{R}$

is called a symmetric bilinear form on E iif

$\forall \lambda, u, v, w \in \mathbb{R} \times E \times E \times E$

$$B(\lambda u + w, v) = \lambda B(u, v) + B(w, v)$$

$$B(u, v) = B(v, u)$$

Def 7.5

$Q : E \longrightarrow \mathbb{R}$

is a positive quadratic form on E iif

$\exists$ a symmetric bilinear form on E such that

$$Q[u] = B[u, u]$$

and $\forall u \in E, \quad Q[u] \geq 0$

Rk: $\quad B[u, v] = \frac{1}{2}\left[Q[u+v] - Q[u] - Q[v]\right]$

Proposition 7.6

If X is a Gaussian vector with values in E
there exists $m_x \in E$ and a positive
quadratic form Q_X on E such that $\forall u \in E$

$$\mathbb{E}[\langle u, X\rangle] = \langle u, m_X\rangle$$

$$\mathrm{Var}[\langle u, X\rangle] = Q_X[u]$$

$$\mathbb{E}\left[\exp(i\langle u, X\rangle)\right] = \exp\left[i\langle u, m_X\rangle - \frac{1}{2} Q_X[u]\right]$$

proof

Let $(e_1, e_2, \dots, e_d)$ be an orthonormal basis of E
and $X = \sum_{j=1}^{d} X_j e_j$ the decomposition of X
in this basis.

The rand. var. X_j are Gaussian

It is easy to check that $m_X = \sum_{j=1}^{d} \mathbb{E}[X_j]\, e_j$ $\Big(= \underset{\text{notation}}{\mathbb{E}[X]}\Big)$

and if $u = \sum_{j=1}^{d} u_j e_j$ $\quad Q_X[u] = \sum_{j,k=1}^{d} u_j u_k \,\mathrm{Cov}(X_j, X_k)$

the Fourier transform follows from the fact that

$$\langle u, X\rangle \text{ is } \mathcal{N}\left[\langle u, m_X\rangle, Q_X[u]\right]$$

Proposition 7.7

Under the previous assumptions, the random variables $X_1, \dots, X_d$ are indep. iif the covariance matrix $(\mathrm{Cov}(X_j, X_k))$ is diagonal, or iif the quadratic form Q_X is diagonal in the basis $(e_1, \dots, e_d)$ $\left(\begin{array}{l} Q_X[u] = B[u,u] \\ B[e_i, e_j] = \lambda_j \delta_{ij} \end{array} \right)$

proof:

$X_1, \dots, X_d$ indep $\Rightarrow$ $(\mathrm{Cov}(X_j, X_k))_{j,k=1,\dots,d}$ diagonal

trivial

If $(\mathrm{Cov}(X_j, X_k))_{j,k}$ is diagonal then

$$Q_X[u] = \sum_{j=1}^{d} \lambda_j u_j^2 \quad \text{where } \lambda_j = \mathrm{Var}[X_j]$$

thus

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^{d} u_j X_j\right)\right] = \prod_{j=1}^{d} \exp\left[i u_j \mathbb{E}[X_j] - \frac{1}{2} \lambda_j u_j^2\right]$$

$$= \prod_{j=1}^{d} \mathbb{E}\left[\exp(i u_j X_j)\right]$$

$\Rightarrow X_1, \dots, X_d$ are indep.

<u>Def</u> γ is a symmetric positive endomorphism on E iif γ is a linear function from E onto E such that $\forall u, v \in E$

$$(u, \gamma[v]) = (\gamma[u], v)$$

$$(u, \gamma[u]) \geqslant 0$$

<u>Rk</u>: Q_x is a quadratic form $\Leftrightarrow$

$\exists$ a symmetric positive endomorphism associated to Q_x /

$\forall u \in E, \quad Q_x[u] = \langle u, \gamma_x[u] \rangle$

The matrix of γ_x in the basis $(e_1, \dots, e_d)$ is $(\mathrm{Cov}(X_j, X_k))$ but the definition of γ_x doesn't depend on the basis

$$\left(\langle v, \gamma_x[u] \rangle = \frac{1}{4} \left[Q_x[u+v] - Q_x[u-v] \right] \right.$$

$\rightarrow$ From now on, to simplify the presentation we will restrict ourselves to centered Gauss vect., i.e. such that $m_x = 0$

Thm 7.8

1) If γ is a symmetric positive endomorphism on E, then there exists a centered Gaussian vector X such that $\gamma_X = \gamma$

2) Let X be a centered Gaussian vector $(\varepsilon_1, \dots, \varepsilon_d)$ a basis of E diagonalizing γ_X

$$\gamma_X \varepsilon_j = \lambda_j \varepsilon_j \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \dots \lambda_r > 0 = \lambda_{r+1} = \dots = \lambda_d$$

(r is the rank of γ_X)

Then $X = \sum_{j=1}^{r} Y_j \varepsilon_j$

where the random variables Y_j are indep. centered Gaussian rand. var. of variance λ_j

If $\mathbb{P}_X$ stands for the law of X, the topological support of $\mathbb{P}_X$ is span $(\varepsilon_1, \dots, \varepsilon_r)$

Ex

Let $X = (X_1, X_2, X_3)$ be a centered Gaussian vector with covariance matrix

$$K = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

a) What is the law of $X_2 + X_3$?

b) Are $X_2 + X_3$ & X_1 independent

c) Are $X_2 + X_3$ & $X_2 - X_3$ independent

d) What is $\mathbb{E}[X_1 | X_2, X_3]$?

e) What is the conditional law of X_1 conditioned on (X_2, X_3) ?

Sol

a) $V[X_2 + X_3] = V[X_2] + V[X_3] + 2\,\mathrm{Cov}[X_2, X_3]$

$$= (0\ 1\ 1)\, K \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 2 + 2 + 2 = 6$$

$X_2 + X_3$ is $\mathcal{N}(0, 6)$

b) $\mathrm{Cov}(X_1, X_2 + X_3) = (1\ 0\ 0)\, K \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$= \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3)$

$= 0 + 2 \neq 0$

$\Rightarrow$ they are not independent

c) $\mathrm{Cov}(X_2 + X_3, X_2 - X_3) = \mathrm{Var}[X_2] - \mathrm{Var}[X_3]$

$= 2 - 2$

$= 0$

$\Rightarrow$ They are indep.

d) $\mathbb{E}[X_1 | X_2, X_3] = a X_2 + b X_3$

$\mathrm{Cov}(X_1 - a X_2 - b X_3, X_2) = 0$

$\mathrm{Cov}(X_1 - a X_2 - b X_3, X_3) = 0$

$\begin{cases} -2a - b = 0 \\ 2 - a - b = 0 \end{cases}$

$\mathbb{E}[X_1 | X_2, X_3] = -\frac{2}{3} X_2 + \frac{4}{3} X_3$

e) Law of $(X_1 | X_2, X_3) = \mathcal{N}(\mathbb{E}[X_1 | X_2, X_3], \sigma^2)$

$\sigma^2 = \mathrm{Var}[X_1 - (-\frac{2}{3} X_2 + \frac{4}{3} X_3)]$

$= \frac{1}{3}$

Law $(X_1 | X_2, X_3) = \mathcal{N}(-\frac{2}{3} X_2 + \frac{4}{3} X_3, \frac{1}{3})$

7.2 Gaussian spaces and processes

Def 7.9

A (centered) Gaussian space is a closed subspace of $L^2(\Omega, \mathcal{U}, \mathbb{P})$ whose elements are centered Gaussian random variables.

Ex: If $(X_1, \dots, X_d)$ is a centered Gauss. vector on $\mathbb{R}^d$, span $\{X_1, \dots, X_d\}$ is a Gaussian space

Let T be an arbitrary set

Def 7.10

A family $(\{X_t, t \in T\})$ is a (centered) Gaussian process iif $\forall n \in \mathbb{N}$, $\forall t_1, \dots, t_n \in T$

$\forall \lambda_1, \dots, \lambda_n \in \mathbb{R}$

$\lambda_1 X_{t_1} + \dots + \lambda_n X_{t_n}$

is a centered Gaussian random variable

Thm 7.11

Let H be a Gaussian space and let $(H_i, i \in I)$ be a family of closed sub-vector spaces of H.

Then the sub-spaces H_i, $i \in I$ are orthogonal in $\mathbb{L}^2$ $(\forall X_i \in H_i, X_j \in H_j, \ \mathbb{E}[X_i X_j] = 0)$

iif the σ-algebra $\sigma(H_i)$, $i \in I$ are indep.

($\sigma(H_i)$: σ-algebra generated by elements of H_i)
rand. var.

Rk. It is important that the spaces H_i are contained in the same Gauss. space

ex: $X_1 = X \quad X_2 = \varepsilon X \quad \varepsilon$ indep of X

$\mathbb{P}[\varepsilon = 1] = \mathbb{P}[\varepsilon = -1] = \frac{1}{2}$

$$\mathbb{E}[X_1 X_2] = \mathbb{E}[\varepsilon]\ \mathbb{E}[X^2] = 0$$

but X_1 & X_2 are not indep (see $X_1 + X_2$)

Corollary 7.12

Let H be a Gaussian space and K a closed sub-vector space of H

Write p_K the orthogonal projection on K

$$p_K : H \longrightarrow K$$
$$X \longrightarrow p_K(X) = \underset{Z \in K}{\operatorname{argmin}} \mathbb{E}[|Z-X|^2]$$
$$= \{ Z \in K \text{ minimizing } \mathbb{E}[|Z-X|^2] \}$$

Let $X \in H$, we have

(i) $\mathbb{E}[X | \sigma(K)] = p_K(X)$

(ii) Let $\sigma^2 = \mathbb{E}[(X - p_K(X))^2]$ then $\forall \Gamma \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}[X \in \Gamma | \sigma(K)] = \frac{1}{\sigma\sqrt{2\pi}} \int_\Gamma dy \exp\left(- \frac{(Y - \overset{\substack{\text{random} \in K \\ \downarrow}}{p_K(X)})^2}{2\sigma^2}\right)$$

Rk In general

$$\mathbb{E}[X \mid \sigma(K)] = P_{L^2(\Omega, \sigma(K), \mathbb{P})}(X)$$

$$\text{here} \quad = P_K(X)$$

K: much smaller than $L^2(\Omega, \sigma(K), \mathbb{P})$

Ex Let $X = (X_1, X_2, X_3)$ be a Gaussian vector. What is the best approximation of X_3 knowing X_1, X_2 ?

Sol: In general $\mathbb{E}[X_3 \mid X_1, X_2] = f(X_1, X_2)$

From the previous thm we know that

$$\mathbb{E}[X_3 \mid X_1, X_2] = \lambda_1 X_1 + \lambda_2 X_2$$

where λ_1, λ_2 are determined by the fact that

$X_3 - (\lambda_1 X_1 + \lambda_2 X_2)$ is $\underset{\text{orthogonal}}{\perp}$ to span(X_1, X_2)