We are interested in estimating the response function of a Neuron (of a frog for instance). The neuron takes an electric signal $x(t)$ as an input and restitutes an electric signal $y(t)$ as an output given by

$$y(t) = \int_0^t K(t-s)dx(s) + Z_t \tag{1}$$

where $s \to K(s)$ is the linear response kernel that we would like to estimate and $t \to Z_t$ is the stochastic process whose law is unknown and independent of $t \to x(t)$. This problem is not trivial since the mean $t \to Z_t$ is also unknown (and not necessarily equal to 0).

In this exercise we are going to propose a method for estimating the Kernel K.

Let $t \to B_t$ be a Brownian Motion on $\mathbb{R}^+$. So there exists a Gaussian measure G (a mapping from $\mathbb{L}^2(\mathbb{R}^+, dx)$) onto a Gaussian space which is linear and such that for $f, g \in \mathbb{L}^2(\mathbb{R}^+, dx))$, $\mathbb{E}[G(f)G(g)] = \int_0^\infty f(s)g(s)ds$ such that $B_t = G[1_{[0,t]}]$. We remind that we write for $f \in \mathbb{L}^2(\mathbb{R}^+, dx))$

$$\int_0^t f(s)dB_s = G[f 1_{[0,t]}] \tag{2}$$

Assume that $r \to K(r) \in \mathbb{L}^2(\mathbb{R}^+, dx)$

a Define $X_t := \int_0^t K(t-s)dB_s$. Is $(X_t)_{t\in\mathbb{R}^+}$ a Gaussian process? justify your answer. If yes then compute the covariance function $\Gamma(s,t) = \mathbb{E}[X_s, X_t]$.

b Assume that $(Z_t)_{t\in\mathbb{R}^+}$ is a bounded integrable stochastic process independent from $(B_t)_{t\in\mathbb{R}^+}$. Write

$$y_t(B) := \int_0^t K(t-s)dB_s + Z_t \tag{3}$$

Let $f \in L^2([0,t], dx)$, show that

$$\mathbb{E}[y_t(B) \int_0 tf(s)dB_s] = \int_0 tf(s)K(t-s)ds. \tag{4}$$

c Fix $a > 0$ and $f \in L^2([0,t], dx)$. Let $(B_t^i)_{1\le i\le m}$ be m independent Brownian motions. Write

$$W_i := y_a(B^i) \int_0^a f(a-s)dB_s^i. \tag{5}$$

Is it true that for all $p \geq 1$, $\mathbb{E}[(W_i)^p] < \infty$ (justify your answer). Write

$$U_m := \frac{1}{m}\sum_{i=1}^m W_i \tag{6}$$

Is U_m converging almost surely as $m \to \infty$? justify your answer. If yes, give its limit.

d Let's consider our initial problem: the estimation of K when only x and y are observed. Let $(\phi_n)_{n\in N}$ be an orthonormal basis of $L^2([0,a], dx)$. Propose a method for estimating $\int_0^a \phi_n(s)K(s)ds$ when K is unknown but $t \to x(t)$ can be specified to be a Brownian motion and $y_t = \int_0^t K(t-s)dx(s) + Z_t$ can be observed.