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OPTIMAL SANDWICH BEAM DESIGN FOR MAXIMUM VISCOELASTIC DAMPING†

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Abstract—Three-layer sandwich beams, made of two elastic outer layers and a viscoelastic layer sandwiched between them, are considered as damping structural elements. A sixth-order equation of motion with complex coefficients of a sandwich beam in free vibrations is reviewed and solved numerically for a large variety of boundary conditions. The solution is used later as part of an optimal design program. An equality constrained minimization algorithm is modified and used to obtain optimal design of damping sandwhich beams subjected to inequality design constraints. The use of the program is demonstrated by solving two design problems

INTRODUCTION

Structural vibration is a major design problem and in most cases designers try to minimize vibration amplitudes in order to eliminate the danger of fatigue failure. The hazard is usually greater in thin walled structures where low modes occur at relatively low frequencies. One method of decreasing vibration amplitudes of such structures is to use layered viscoelastic materials of high damping properties in such a way, that the stiffness of the structure remains high enough to support the load, whereas the viscoelastic material provides the necessary damping characteristics. The first analysis of such a structure was done by Oberst (1952) on an elastic beam with a viscoelastic layer bonded to one or two of its faces. The efficiency of that structure, however, was not high because the viscoelastic material was subjected to low tension/compression strains along the beam's axis and the shear was relatively low, thus inhibiting the damping property of the viscoelastic material from being realized. A better design is to constrain the viscoelastic layer between stiff elastic layers, forcing it to undergo high shear deformations that are accompanied with high energy losses. Such a design was analyzed by Kerwin (1959), and this was followed by a large number of papers dealing with different aspects of the problem. Some reviews of the work on vibration control with viscoelastic materials have been written by Nakra (1976, 1981) and Nelson (1977).

This paper deals with optimal design of three-layer (not necessarily symmetrical) sandwich beams made of two elastic outer layers and a viscoelastic layer sandwiched between them. Some of the previous work on similar beams is discussed here.

Most of the papers on this subject end up with an equation of motion of the sixth order, with complex coefficients (Ditaranto, 1965; Ditaranto and Blasingame, 1967; Mead and Markus, 1969), which is then solved for certain boundary conditions. The solution is quite complex and only in 1978 did Rao (1978) manage to get a numerical solution for a wide range of boundary conditions.

The damping of the beam (η) is generally plotted as a function of (the real part of) a shear parameter g given by

$$g = \frac{G_2(1+i\eta_2)L^2 b(E_1A_1+E_3A_3)}{H_2E_1A_1E_3A_3} \tag{1}$$

for different materials and cross-sectional configuration designated by a geometric parameter Y, where

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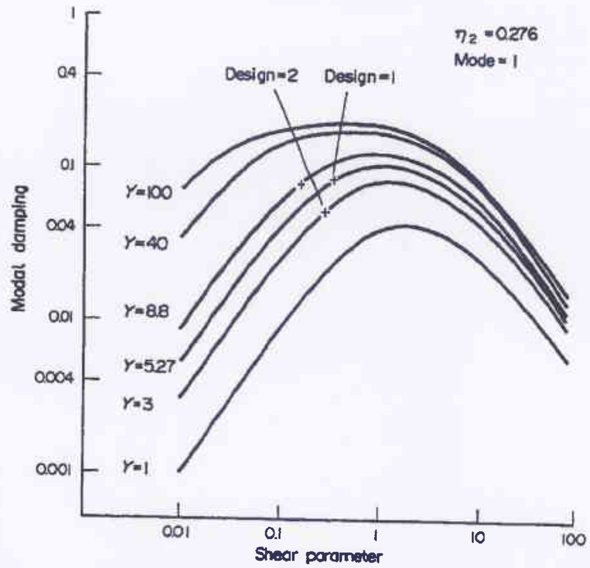

Fig. 1. Variation of sandwich beam damping with shear parameter for various Ys.

$$Y = (d^2/D_t)E_1A_1E_3A_3/(E_1A_1+E_3A_3). \tag{2}$$

A typical plot of the damping of a cantilever beam vibrating in its first mode is shown in Fig. 1. It is clear that the damping of the sandwich beam goes through a maximum, but it is not obvious how to get this maximum damping. The rate dependence of the shear modulus of the viscoelastic material complicates even more the problem of selecting a material and cross-sectional geometry for maximum damping.

The purpose of the present work is to give the designer an automated tool that will enable him to select the proper materials and cross-sectional geometry of the beam such that the modal damping is a maximum under predetermined design constraints. The present program handles a variety of boundary conditions and design constraints and can be used as an interactive program.

The next section presents a short description of the model, the equation of motion, boundary conditions, and the type of solution sought. This is followed by a section on the numerical solution and optimal design. In the last section two design concepts are illustrated using the present optimization program.

GOVERNING EQUATIONS OF THE PROBLEM

The equation of motion for transverse vibrations and the solution type, follows the derivation given by Mead and Markus (1969). Consider a sandwich beam made of three layers: two elastic face-plates with thicknesses H_1 and H_3 and moduli E_1 and E_3, respectively, and a viscoelastic core of thickness H_2, density ρ_2, and complex shear modulus $G = G_2(1+i\eta_2)$. The width of the beam is b, and its length L. The assumptions that lead to the equation of motion are:

(a) the elastic face-plates carry only longitudinal stresses;
(b) the core carries only shear stresses and is modelled as a linear viscoelastic material;
(c) transverse strains are neglected in both core and face-plates;
(d) the layers are perfectly bonded;
(e) the longitudinal and rotatory inertia are neglected.

The equation of motion for the transverse displacement $w(x, t)$ resulting from an externally time-dependent loading $q(x, t)$ is

$$\frac{\partial^6 w}{\partial \bar{x}^6} - g(1+Y)\frac{\partial^4 w}{\partial \bar{x}^4} + \frac{\partial^4 w}{\partial \bar{x}^2\,\partial \bar{t}^2} - g\frac{\partial^2 w}{\partial \bar{t}^2} = \frac{L^4}{D_t}\left(\frac{\partial^2 q}{\partial \bar{x}^2} - gq\right) \tag{3}$$

where $\bar{x}$ and $\bar{t}$ are normalized length and time

$$\bar{x} = x/L \quad \text{and} \quad \bar{t} = t/t_0; \quad [t_0 = (mL^4/D_t)^{1/2}]$$

m is the mass of the beam per unit length and D_t is the total flexural rigidity

$$D_t = b(E_1 H_1^3 + E_3 H_3^3)/12. \tag{4}$$

Mead and Markus then considered harmonic motion

$$w(\bar{x}, \bar{t}) = W_n(\bar{x})T(\omega, \bar{t}) \tag{5}$$

produced by harmonically varying load, q, which is proportional to the nth mode $W_n(\bar{x})$

$$q(\bar{x}, \bar{t}) = PmW_n(\bar{x})\ \mathrm{e}^{\mathrm{i}\omega\bar{t}t_0}. \tag{6}$$

Substitution of eqns (5) and (6) in eqn (3) leads to the two equations

$$\ddot{T} + \bar{\omega}_n^2(1+\mathrm{i}\eta_n)T = Pt_0^2\ \mathrm{e}^{\mathrm{i}\bar{\omega}\bar{t}} \tag{7}$$

and

$$W_n^{\mathrm{VI}} - g(1+Y)W_n^{\mathrm{IV}} - \bar{\omega}_n^2(1+\mathrm{i}\eta_n)(W_n^{\mathrm{II}} - gW_n) = 0 \tag{8}$$

where the second time derivative of T is denoted by $\ddot{T}$; the x-wise derivatives are denoted by $W^{\mathrm{I}}, W^{\mathrm{II}}, \ldots, W^{\mathrm{VI}}$; and $\bar{\omega}$ is the normalized angular frequency of the applied load ($\bar{\omega} = \omega t_0$).

The first step toward the design of an optimal sandwich beam is to develop a numerical scheme for determining the values of the (normalized) natural frequencies, $\bar{\omega}_n$, and modal damping, η_n, of a given sandwich beam subjected to various boundary conditions. The second step is to establish some design constraints and develop a numerical algorithm that determines the dimensions and mechanical properties of the layers such that the damping of the sandwich beam is maximum.

The general solution of eqn (8) is of the form

$$W_n = \sum_{s=1}^{6} A_{ns}\ \mathrm{e}^{\lambda_{ns}\bar{x}} \tag{9}$$

which leads to the auxiliary equation for each of the six complex roots λ_{ns}

Table 1. Basic boundary conditions

Boundary condition	Notation	Detailed boundary conditions 1	2	3
Free	F	$W^{\mathrm{II}} = 0$	$W^{\mathrm{IV}} - W\bar{\omega}_n^2(1+\mathrm{i}\eta_n) = 0$	$W^{\mathrm{V}} - g(1+Y)W^{\mathrm{III}} - W^{\mathrm{I}}\bar{\omega}_n^2(1+\mathrm{i}\eta_n) = 0$
Free-riveted	$\mathrm{F_r}$	$W^{\mathrm{III}} = 0$	$W^{\mathrm{V}} - W^{\mathrm{I}}\bar{\omega}_n^2(1+\mathrm{i}\eta_n) = 0$	$W^{\mathrm{IV}} - g(1+Y)W^{\mathrm{II}} - W\bar{\omega}_n^2(1+\mathrm{i}\eta_n) = 0$
Pinned	P	$W = 0$	$W^{\mathrm{II}} = 0$	$W^{\mathrm{IV}} = 0$
Pinned-riveted	$\mathrm{P_r}$	$W = 0$	$W^{\mathrm{IV}} - g(1+Y)W^{\mathrm{II}} = 0$	$W^{\mathrm{V}} - gYW^{\mathrm{III}} - W^{\mathrm{I}}\bar{\omega}_n^2(1+\mathrm{i}\eta_n) = 0$
Clamped-allows long. sliding	$\mathrm{C_u}$	$W = 0$	$W^{\mathrm{I}} = 0$	$W^{\mathrm{IV}} - gYW^{\mathrm{II}} = 0$
Clamped	C	$W = 0$	$W^{\mathrm{I}} = 0$	$W^{\mathrm{V}} - gYW^{\mathrm{III}} = 0$
Sliding	S	$W^{\mathrm{I}} = 0$	$W^{\mathrm{IV}} - gYW^{\mathrm{II}} - W\bar{\omega}_n^2(1+\mathrm{i}\eta_n) = 0$	$W^{\mathrm{V}} - g(1+Y)W^{\mathrm{III}} = 0$
Sliding-riveted	$\mathrm{S_r}$	$W^{\mathrm{I}} = 0$	$W^{\mathrm{III}} = 0$	$W^{\mathrm{V}} = 0$

$$\lambda_{ns}^6 - g(1+Y)\lambda_{ns}^4 - \bar{\omega}_n^2(1+\mathrm{i}\eta_n)(\lambda_{ns}^2 - g) = 0. \tag{10}$$

The boundary conditions that have been considered are summarized in Table 1, whereas Table 2 shows the combinations of these boundary conditions that are available in our program. Substitution of the general solution (9) in the boundary conditions of a given beam leads to a set of six homogeneous complex equations in A_{ns}, with coefficients that are functions of the (yet unknown) λ_{ns}, $\bar{\omega}_n$ and η_n. Using the relations between the roots of cubic equations (Mead and Markus, 1970), and considering eqn (9) to be a cubic in λ_n^2, one can express its roots as a function of, say, λ_{n1}. From eqn (9) the natural frequency and modal damping can also be expressed as a function of λ_{n1}

$$\bar{\omega}_n^2(1+\mathrm{i}\eta_n) = \lambda_{n1}^4(1 - Yg/(\lambda_{n1}^2 - g)). \tag{11}$$

Thus, the determinant derived from the boundary conditions is expressed as a function of a single complex root λ_{n1}. This determinant is solved numerically, following the procedure developed by Rao (1977), by using an improved iteration procedure with complex double precision. The starting value of λ_{n1} is that of a corresponding Euler beam.

AUTOMATED OPTIMAL DESIGN

The procedure outlined in the previous section is suitable for calculating natural frequencies and damping of a given beam. This, however, is not the problem facing a design engineer who wants to design a structure (beam) for maximum damping. What he is facing is a number of design constraints derived from considerations such as: weight, rigidity, total thickness, material properties, etc. Within the boundary of the constraints he has to

Table 2. Boundary conditions available in present program

No.	Left	Right
1	C	F
2	C	$\mathrm{F_r}$
3	C	P
4	C	$\mathrm{P_r}$
5	C	C
6	$\mathrm{P_r}$	P
7	$\mathrm{P_r}$	$\mathrm{P_r}$
8	P	P
9	F	F
10	F	$\mathrm{F_r}$
11	$\mathrm{F_r}$	$\mathrm{F_r}$

select the appropriate cross-sectional geometry and materials such that the damping is maximum.

Markus *et al.* (1974) used nomograms to design a symmetrical sandwich beam with an optimal shear parameter g (i.e. to have damping corresponding to the peak of a given curve in Fig. 1). In their example they designed a symmetrical sandwich beam subject to one equality constraint namely, weight ratio equal to one. It is not clear to us how effective their method is in the presence of a number of inequality constraints.

Rao (1978) presented an "optimal design example" of a symmetrical sandwich beam subject to three equality constraints: weight, height, and $Y = 15$. He assumed that all but four parameters (H_1, H_2, ρ_2, and G_2) were given, so that the three constraints were used to calculate three parameters (H_1, H_2, and ρ_2). The "optimization" was to move along the curve $Y = 15$ to its peak, in a plot similar to Fig. 1, and select the value of the shear parameter at the peak. Once g is known, the value of the shear modulus G_2 of the damping material can be calculated from eqn (1).

We feel that the geometrical parameter is not a good design constraint because it has no apparent engineering meaning. In fact, by eliminating this constraint one may end up with higher damping values. We also feel that constraints of inequality are more appropriate to use in engineering design.

In the present work we use the algorithm developed by Vardi (1985) for equality constrained minimization. The constraints in our work which are of the type: $h_i(x_j) \leqslant 0$, are converted in the numerical program to the form $h_i(x_j) - r_i^2 = 0$ to fit Vardi's work. The constraints in our program are given by normalizing the appropriate sandwich parameters with respect to those of an equivalent homogeneous beam. The available constraints are:

(a) Weight constraint: $W_{\text{low}} \leqslant W_{\text{sand}}/W_{\text{beam}} \leqslant W_{\text{up}}$, where W_{sand} and W_{beam} are the weights of the sandwich beam and the homogeneous beam respectively; and W_{low} and W_{up} are the prescribed lower and upper bounds on the weight ratio.

(b) Rigidity constraint: $D_{\text{low}} \leqslant D_{\text{sand}}/D_{\text{beam}} \leqslant D_{\text{up}}$, where D_{sand} and D_{beam} are the flexural rigidities of the two beams.

(c) Height constraint: $H_{\text{low}} \leqslant H_{\text{sand}}/H_{\text{beam}} \leqslant H_{\text{up}}$, where H_{sand} and H_{beam} are the total cross-sectional heights of the two beams.

(d) Elastic layer thickness constraint: $H_{1\,\text{low}} \leqslant H_1/H_{\text{beam}} \leqslant H_{1\,\text{up}}$, where H_1 is the thickness of one of the elastic layers of the sandwich beam. The thickness of the other is taken to be equal to H_{beam}.

(e) Damping layer thickness constraint: $H_{2\,\text{low}} \leqslant H_2/H_{\text{beam}} \leqslant H_{2\,\text{up}}$, where H_2 is the thickness of the viscoelastic layer.

(f) Geometric parameter constraint: $Y \geqslant Y_{\text{min}}$. This constraint is introduced into our program in spite of a previous comment, in order to compare our work to the work of others who use Y as an important design parameter. The value of Y_{min} is usually taken to be equal to 15. Note, however, that we allow Y to be equal to or larger than Y_{min}.

The calculation procedure is indicated in Fig. 2. The problem is defined in the "MAIN" program, and this includes the definition of boundary conditions and constraints, defining a homogeneous beam and guessing initial values for the sandwich beam. The next step is to proceed to the optimization program where the values of the sandwich beam are changed by an iteration process such that the damping is maximum. In each iteration cycle the determinant (derived from the boundary conditions) is solved numerically, thus producing a value for the modal damping η_n. The values of the sandwich beam are also checked for conformity with the constraints. This procedure continues until a sandwich beam is found such that its damping is maximum and it meets all the imposed constraints.

RESULTS AND DISCUSSION

To demonstrate the use of this program it was applied to two design problems. Design 1: replace a given homogeneous beam by a three-layer sandwich beam subject to a given set of constraints, and Design 2: add a constrained viscoelastic layer to the given homogeneous beam, subject to a given set of constraints. In both cases the goal is to achieve

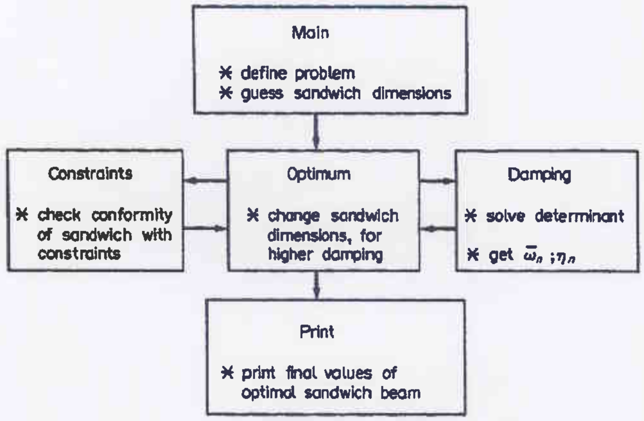

Fig. 2. Schematic description of calculation procedure.

maximum damping of the clamped–free sandwich beam. The values of the homogeneous aluminium beam are:

length,	$L = 180\,\text{mm}$
height,	$H_{\text{beam}} = 5\,\text{mm}$
Young's modulus,	$E_{\text{beam}} = 71\,\text{GPa}$
density,	$\rho_{\text{beam}} = 2700\,\text{kg}\,\text{mm}^{-3}$.

The elastic layers of the sandwich beam are made of the same material, and have the same length and width, as the homogeneous beam. The viscoelastic core material is Neoprene CR-602, characterized by rate-dependent shear modulus and damping

$$G_2(f) = 1.007 \times 10^{-3} f + 1.386\,\text{MPa}$$
$$\eta_2(f) = 1.608 \times 10^{-4} f + 0.2564$$

where f is the frequency in hertz. The constraints for each of the two designs were chosen arbitrarily and do not limit the use of the program in any way.

Design 1: the following constraints apply:

$$1 \cdot a \quad 0.8 \leqslant H_{\text{sand}}/H_{\text{beam}} \leqslant 2$$
$$1 \cdot b \quad 0.8 \leqslant D_{\text{sand}}/D_{\text{beam}} \leqslant 1.2$$
$$1 \cdot c \quad 0.24 \leqslant H_2/H_{\text{beam}}.$$

When the first guess of the thickness was $H_1 = 9$ mm, $H_3 = 7$ mm, and $H_2 = 1$ mm, the program ran through 18 iterations before reaching the optimal design. Final and some

Table 3. Intermediate and final values in design 1 of sandwich beam

Iteration No.	Geometry (mm) H_1	H_2	H_3	$\frac{H_{\text{sand}}}{H_{\text{beam}}}$	$\frac{D_{\text{sand}}}{D_{\text{beam}}}$	Damping η	Frequency (Hz)
0	9	1	7	3.4	8.576	0.0452	220
2	5.37	0.91	3.68	1.992	1.637	0.0695	
6	3.948	0.364	3.958	1.654	0.988	0.0881	
8	3.948	0.366	3.953	1.653	0.986	0.0881	
12	3.670	1.199	3.669	1.708	0.790	0.0820	
14	3.684	1.200	3.684	1.714	0.800	0.0818	
18	3.684	1.200	3.684	1.714	0.800	0.0818	105

Table 4. Intermediate and final values in design 2 of sandwich beam

Iteration No.	Geometry (mm) H_1	H_2	H_3	$\frac{H_{sand}}{H_{beam}}$	$\frac{W_{sand}}{W_{beam}}$	Damping η	Frequency (Hz)
0	4	3	5	2.4	2.088	0.0597	125
2	1.527	0.767	5	1.459	1.379	0.0572	
6	1.673	0.675	5	1.470	1.400	0.0601	
10	1.047	2.999	5	1.809	1.498	0.0522	
12	1.051	3.000	5	1.810	1.499	0.0522	
16	1.055	3.000	5	1.810	1.500	0.0523	124

intermediate values are shown in Table 3. The global convergence of the program was checked by changing the values of the first guess. The final results were the same in all cases and the only difference was the number of iterations prior to reaching the optimal design values.

When some of the constraints are relaxed, the number of iterations is reduced considerably and the value of the damping is increased, as can be seen in Table 3 if we eliminate, say, constraint $1 \cdot c$. In that case we see that after six iterations the damping is already higher than the final value of 0.0818, and the other two constraints are met.

The value of the geometrical parameter Y for this design is $Y = 5.27$. An attempt to add a constraint of $Y \geqslant 15$ to design 1 led to conflicting constraints which prevented the convergence of the solution. The maximum value of Y that is compatible with the other constraints is about $Y = 8.8$, and the associated optimal damping is $\eta = 0.0765$, which is lower than the optimal damping of the original design 1. The optimal values of these two cases as well as the value of design 2 are shown in Fig. 1.

Design 2: The constraints in this example were chosen to be

$$2 \cdot a \quad 1.05 \leqslant W_{sand}/W_{beam} \leqslant 1.5$$

$$2 \cdot b \quad 1.3 \leqslant H_{sand}/H_{beam} \leqslant 2.1$$

$$2 \cdot c \quad 0.2 \leqslant H_1/H_{beam}$$

$$2 \cdot d \quad 0.6 \leqslant H_2/H_{beam}.$$

Final and some intermediate values are shown in Table 4. The value of the geometrical parameter for this design is $Y = 3$, and an attempt to increase this value (while keeping the remaining constraints unchanged) would lead to a lower value of the modal damping, as in the previous design example. It should be clear that it would have been prohibitively difficult to solve the two design examples without an optimization program.

CONCLUSIONS

An automated optimization numerical program has been developed for designing three-layer sandwich beams for maximum damping. The program handles a large variety of boundary conditions and inequality constraints, and it converges rather rapidly even when the initial (guessed) values are very far from the final optimal values. The constraints are based on design requirements, and not on the geometrical parameter Y, which has no apparent engineering significance. Although the use of the program has been demonstrated here with two examples only, many more cases have been solved which included other boundary conditions, different materials for the constraining elastic layer and other combinations of constraints.

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