CHAPTER

8 Counting Principles; Further Probability Topics

If you have 31 ice cream flavors available, how many different three-scoop cones can you make? The answer, which is surprisingly large, involves counting permutations or combinations, the subject of the first two sections in this chapter. The counting formulas we will develop have important applications in probability theory.

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In this chapter, we continue our discussion of probability theory. To use the basic definition of probability, $P(E) = n(E)/n(S)$ (where S is the sample space with equally likely outcomes), up to now we have simply listed the outcomes in S and in E. However, when S has many outcomes, listing them all becomes very tedious. In the first two sections of this chapter, we introduce methods for counting the number of outcomes in a set without actually listing them, and then we use this approach in the third section to find probabilities. In the section on binomial probability (repeated independent trials of an experiment with only two possible outcomes), we introduce a formula for finding the probability of a certain number of successes in a number of trials. The final section continues the discussion of probability distributions that we began in Chapter 7.

8.1 THE MULTIPLICATION PRINCIPLE; PERMUTATIONS

THINK ABOUT IT *In how many ways can seven panelists be seated in a row of seven chairs?*

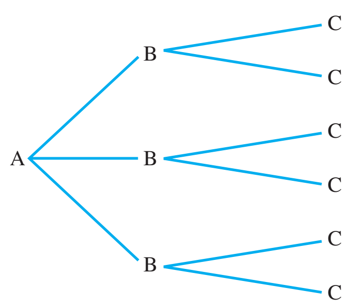

FIGURE 1

This question will be answered later in this section using *permutations.* Let us begin with a simple example. If there are 3 roads from town A to town B and 2 roads from town B to town C, in how many ways can a person travel from A to C by way of B? For each of the 3 roads from A there are 2 different routes leading from B to C, or a total of $3 \cdot 2 = 6$ different ways for the trip, as shown in Figure 1. This example illustrates a general principle of counting, called the **multiplication principle.**

MULTIPLICATION PRINCIPLE

Suppose n choices must be made, with

$$m_1 \text{ ways to make choice 1,}$$

and for each of these ways,

$$m_2 \text{ ways to make choice 2,}$$

and so on, with

$$m_n \text{ ways to make choice } n.$$

Then there are

$$m_1 \cdot m_2 \cdot \cdots \cdot m_n$$

different ways to make the entire sequence of choices.

EXAMPLE 1 *Combination Lock*

A certain combination lock can be set to open to any one 3-letter sequence. How many such sequences are possible?

Solution Since there are 26 letters in the alphabet, there are 26 choices for each of the 3 letters. By the multiplication principle, there are $26 \cdot 26 \cdot 26 = 17{,}576$ different sequences.

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EXAMPLE 2 *Morse Code*

Morse code uses a sequence of dots and dashes to represent letters and words. How many sequences are possible with at most 3 symbols?

Solution "At most 3" means "1 or 2 or 3" here. Each symbol may be either a dot or a dash. Thus the following number of sequences are possible in each case.

Number of Symbols	Number of Sequences
1	$\mathbf{2}$
2	$\mathbf{2} \cdot \mathbf{2} = 4$
3	$\mathbf{2} \cdot \mathbf{2} \cdot \mathbf{2} = 8$

Altogether, $2 + 4 + 8 = 14$ different sequences are possible.

EXAMPLE 3 *I Ching*

FIGURE 2

An ancient Chinese philosophical work known as the *I Ching (Book of Changes)* is often used as an oracle from which people can seek and obtain advice. The philosophy describes the duality of the universe in terms of two primary forces: *yin* (passive, dark, receptive) and *yang* (active, light, creative). See Figure 2. The yin energy is represented by a broken line (– –) and the yang by a solid line (—). These lines are written on top of one another in groups of three, known as *trigrams*. For example, the trigram ☱ is called *Tui*, the Joyous, and has the image of a lake.

(a) How many trigrams are there altogether?

Solution Think of choosing between the 2 types of lines for each of the 3 positions in the trigram. There will be 2 choices for each position, so there are $2 \cdot 2 \cdot 2 = 8$ different trigrams.

(b) The trigrams are grouped together, one on top of the other, in pairs known as *hexagrams*. Each hexagram represents one aspect of the *I Ching* philosophy. How many hexagrams are there?

Solution For each position in the hexagram there are 8 possible trigrams, giving $8 \cdot 8 = 64$ hexagrams.

EXAMPLE 4 *Books*

A teacher has 5 different books that he wishes to arrange side by side. How many different arrangements are possible?

Solution Five choices will be made, one for each space that will hold a book. Any of the 5 books could be chosen for the first space. There are 4 choices for the second space, since 1 book has already been placed in the first space; there are 3 choices for the third space, and so on. By the multiplication principle, the number of different possible arrangements is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

FOR REVIEW

The natural numbers, also referred to as the positive integers, are the numbers 1, 2, 3, 4, etc.

The use of the multiplication principle often leads to products such as $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, the product of all the natural numbers from 5 down to 1. If n is a natural number, the symbol $n!$ (read "*n factorial*") denotes the product of all the

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natural numbers from n down to 1. If $n = 1$, this formula is understood to give $1! = 1$.

FACTORIAL NOTATION

For any natural number n,

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

Also,

$$0! = 1.$$

With this symbol, the product $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can be written as $5!$. Also, $3! = 3 \cdot 2 \cdot 1 = 6$. The definition of $n!$ could be used to show that $n[(n-1)]! = n!$ for all natural numbers $n \geq 2$. It is helpful if this result also holds for $n = 1$. This can happen only if $0!$ equals 1, as defined above.

Some calculators have an $n!$ key. A calculator with a 10-digit display and scientific notation capability will usually give the exact value of $n!$ for $n \leq 13$, and approximate values of $n!$ for $14 \leq n \leq 69$. The value of $70!$ is approximately $1.198 \cdot 10^{100}$, which is too large for most calculators. To see how large $70!$ is, suppose a computer counted the numbers from 1 to $70!$ at a rate of 1 billion numbers per second. If the computer started when the universe began, by now it would only be done with a tiny fraction of the total.

On many graphing calculators, the factorial of a number is accessible through a menu. On the TI-83/84 Plus, for example, this menu is found by pressing the `MATH` key, and then selecting `PRB` (for probability).

EXAMPLE 5 *Books*

Suppose the teacher in Example 4 wishes to place only 3 of the 5 books on his desk. How many arrangements of 3 books are possible?

Solution The teacher again has 5 ways to fill the first space, 4 ways to fill the second space, and 3 ways to fill the third. Since he wants to use only 3 books, only 3 spaces can be filled (3 events) instead of 5, for $5 \cdot 4 \cdot 3 = 60$ arrangements.

Permutations

The answer 60 in Example 5 is called the number of *permutations* of 5 things taken 3 at a time. A **permutation** of r (where $r \geq 1$) elements from a set of n elements is any specific ordering or arrangement, *without repetition,* of the r elements. Each rearrangement of the r elements is a different permutation. The number of permutations of n things taken r at a time (with $r \leq n$) is written $P(n, r)$. Based on the work in Example 5,

$$P(5, 3) = 5 \cdot 4 \cdot 3 = 60.$$

Factorial notation can be used to express this product as follows.

$$5 \cdot 4 \cdot 3 = 5 \cdot 4 \cdot 3 \cdot \frac{2 \cdot 1}{2 \cdot 1} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

This example illustrates the general rule of permutations, which can be stated as follows.

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PERMUTATIONS

If $P(n, r)$ (where $r \leq n$) is the number of permutations of n elements taken r at a time, then

$$P(n, r) = \frac{n!}{(n-r)!}.$$

CAUTION The letter P here represents *permutations,* not *probability.* In probability notation, the quantity in parentheses describes an *event.* In permutations notation, the quantity in parentheses always comprises *two numbers.* ■

The proof of the permutations rule follows the discussion in Example 5. There are n ways to choose the first of the r elements, $n - 1$ ways to choose the second, and $n - r + 1$ ways to choose the rth element, so that

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1).$$

Now multiply on the right by $(n - r)!/(n - r)!$.

$$\begin{aligned} P(n, r) &= n(n-1)(n-2)\cdots(n-r+1)\cdot\frac{(n-r)!}{(n-r)!} \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

To find $P(n, r)$, we can use either the permutations formula or direct application of the multiplication principle, as the following example shows.

EXAMPLE 6 *Politics*

By the end of September 2003, ten candidates sought the Democratic nomination for president. In how many ways could voters rank their first, second, and third choices?

Solution

Method 1: Calculating by Hand

This is the same as finding the number of permutations of 10 elements taken 3 at a time. Since there are 3 choices to be made, the multiplication principle gives $P(10, 3) = 10 \cdot 9 \cdot 8 = 720$. Alternatively, use the permutations formula to get

$$P(10, 3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720.$$

Method 2: Graphing Calculator

Graphing calculators have the capacity to compute permutations. For example, on a TI-83/84 Plus, $P(10, 3)$ can be calculated by inputting 10 followed by `nPr` (found in the `MATH-PRB` menu), and a 3 yielding 720, as shown in Figure 3.

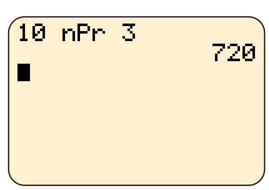

FIGURE 3

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Method 3: Spreadsheet Spreadsheets can also compute permutations. For example, in Microsoft Excel, $P(10, 3)$ can be calculated by inputting 10 and 3 in cells, say, A1 and B1, and then typing "`=FACT(A1)/FACT(A1-B1)`" in cell C1 or, for that matter, any other cell.

CAUTION When calculating the number of permutations with the formula, do not try to cancel unlike factorials. For example,

$$\frac{8!}{4!} \neq 2! = 2 \cdot 1 = 2.$$

$$\frac{8!}{4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680.$$

Always write out the factors first, then cancel where appropriate. ■

EXAMPLE 7 *Permutations*

Find the following.

(a) The number of permutations of the letters A, B, and C

Solution By the formula for $P(n, r)$ with both n and r equal to 3,

$$P(3, 3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3!}{1} = 3 \cdot 2 \cdot 1 = 6.$$

The 6 permutations (or arrangements) are

ABC, ACB, BAC, BCA, CAB, CBA.

(b) The number of permutations if just 2 of the letters A, B, and C are to be used

Solution Find $P(3, 2)$.

$$P(3, 2) = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3! = 6$$

This result is exactly the same answer as in part (a). This is because, in the case of $P(3, 3)$, after the first 2 choices are made, the third is already determined, as shown in the table below.

First Two Letters	AB	AC	BA	BC	CA	CB
Third Letter	C	B	C	A	B	A

EXAMPLE 8 *Television*

A televised talk show will include 4 women and 3 men as panelists.

(a) In how many ways can the panelists be seated in a row of 7 chairs?

Solution Find $P(7, 7)$, the total number of ways to seat 7 panelists in 7 chairs.

$$P(7, 7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

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There are 5040 ways to seat the 7 panelists.

(b) In how many ways can the panelists be seated if the men and women are to be alternated?

Solution Use the multiplication principle. In order to alternate men and women, a woman must be seated in the first chair (since there are 4 women and only 3 men), any of the men next, and so on. Thus there are 4 ways to fill the first seat, 3 ways to fill the second seat, 3 ways to fill the third seat (with any of the 3 remaining women), and so on. This gives

$$4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 144$$

ways to seat the panelists.

(c) In how many ways can the panelists be seated if the men must sit together, and the women must also sit together?

Solution Use the multiplication principle. We first must decide how to arrange the two groups (men and women). There are 2! ways of doing this. Next, there are 4! ways of arranging the women and 3! ways of arranging the men, for a total of

$$2!\,4!\,3! = 2 \cdot 24 \cdot 6 = 288$$

ways.

(d) In how many ways can one woman and one man from the panel be selected?

Solution There are 4 ways to pick the woman and 3 ways to pick the man, for a total of

$$4 \cdot 3 = 12$$

ways.

If the n objects in a permutation are not all distinguishable—that is, if there are n_1 of type 1, n_2 of type 2, and so on for r different types, then the number of **distinguishable permutations** is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

For example, suppose we want to find the number of permutations of the numbers 1, 1, 4, 4, 4. We cannot distinguish between the two 1's or among the three 4's, so using 5! would give too many distinguishable arrangements. Since the two 1's are indistinguishable and account for 2! of the permutations, we divide 5! by 2!. Similarly, we also divide by 3! to account for the three indistinguishable 4's. This gives

$$\frac{5!}{2!\,3!} = 10$$

permutations.

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EXAMPLE 9 *Mississippi*

In how many ways can the letters in the word *Mississippi* be arranged?

Solution This word contains 1 m, 4 i's, 4 s's, and 2 p's. To use the formula, let $n = 11$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, and $n_4 = 2$ to get

$$\frac{11!}{1!\,4!\,4!\,2!} = 34{,}650$$

arrangements.

NOTE If Example 9 had asked for the number of ways that the letters in a word with 11 *different* letters could be arranged, the answer would be $11! = 39{,}916{,}800$.

EXAMPLE 10 *Yogurt*

A student buys 3 cherry yogurts, 2 raspberry yogurts, and 2 blueberry yogurts. She puts them in her dormitory refrigerator to eat one a day for the next week. Assuming yogurts of the same flavor are indistinguishable, in how many ways can she select yogurts to eat for the next week?

Solution This problem is again one of distinguishable permutations. The 7 yogurts can be selected in 7! ways, but since the 3 cherry, 2 raspberry, and 2 blueberry yogurts are indistinguishable, the total number of distinguishable orders in which the yogurts can be selected is

$$\frac{7!}{3!\,2!\,2!} = 210.$$

8.1 EXERCISES

In Exercises 1–12, evaluate the factorial or permutation.

1. 6!
2. 7!
3. 15!
4. 16!
5. $P(13, 2)$
6. $P(12, 3)$
7. $P(38, 17)$
8. $P(33, 19)$
9. $P(n, 0)$
10. $P(n, n)$
11. $P(n, 1)$
12. $P(n, n - 1)$

13. How many different types of homes are available if a builder offers a choice of 5 basic plans, 3 roof styles, and 2 exterior finishes?

14. A menu offers a choice of 3 salads, 8 main dishes, and 5 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

15. A couple has narrowed down the choice of a name for their new baby to 3 first names and 5 middle names. How many different first- and middle-name arrangements are possible?

16. In a club with 15 members, how many ways can a slate of 3 officers consisting of president, vice-president, and secretary/treasurer be chosen?

17. Define *permutation* in your own words.

18. In Example 7, there are six 3-letter permutations of the letters A, B, and C. How many 3-letter subsets (unordered groups of letters) are there?

19. In Example 7, how many unordered 2-letter subsets of the letters A, B, and C are there?

20. Find the number of distinguishable permutations of the letters in each word.

a. initial **b.** little **c.** decreed

21. A printer has 5 A's, 4 B's, 2 C's, and 2 D's. How many different "words" are possible that use all these letters? (A "word" does not have to have any meaning here.)

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22. Wing has different books to arrange on a shelf: 4 blue, 3 green, and 2 red.

a. In how many ways can the books be arranged on a shelf?

b. If books of the same color are to be grouped together, how many arrangements are possible?

c. In how many distinguishable ways can the books be arranged if books of the same color are identical but need not be grouped together?

d. In how many ways can you select 3 books, one of each color, if the order in which the books are selected does not matter?

23. A child has a set of differently shaped plastic objects. There are 3 pyramids, 4 cubes, and 7 spheres.

a. In how many ways can she arrange the objects in a row if each is a different color?

b. How many arrangements are possible if objects of the same shape must be grouped together and each object is a different color?

c. In how many distinguishable ways can the objects be arranged in a row if objects of the same shape are also the same color, but need not be grouped together?

d. In how many ways can you select 3 objects, one of each shape, if the order in which the objects are selected does not matter and each object is a different color?

24. Some students find it puzzling that $0! = 1$, and think that $0!$ should equal 0. If this were true, what would be the value of $P(4, 4)$ using the permutations formula?

25. If you already knew the value of 9!, how could you find the value of 10! quickly?

26. When calculating $n!$, the number of ending zeros in the answer can be determined prior to calculating the actual number by finding the number of times 5 can be factored from $n!$. For example, 7! only has one 5 occurring in its calculation, and so there is only one ending zero in 5040. The number 10! has two 5's (one from the 5 and one from the 10) and so there must be two ending zeros in the answer 3,628,800. Use this idea to determine the number of zeros that occur in the following factorials, and then explain why this works.

a. 13! **b.** 27! **c.** 75!

27. Because of the view screen, calculators only show a fixed number of digits, often 10 digits. Thus, an approximation of a number will be shown by only including the 10 largest place values of the number. Using the ideas from the previous exercise, determine if the following numbers are correct or if they are incorrect by checking if they have the correct number of ending zeros. (*Note:* Just because a number has the correct number of zeros does not imply that it is correct.)

a. $12! = 479{,}001{,}610$ **b.** $23! = 25{,}852{,}016{,}740{,}000{,}000{,}000{,}000$

c. $15! = 1{,}307{,}643{,}680{,}000$ **d.** $14! = 87{,}178{,}291{,}200$

Applications

BUSINESS AND ECONOMICS

28. ***Automobile Manufacturing*** An automobile manufacturer produces 7 models, each available in 6 different exterior colors, with 4 different upholstery fabrics and 5 interior colors. How many varieties of automobile are available?

LIFE SCIENCES

29. ***Drug Sequencing*** Eleven drugs have been found to be effective in the treatment of a disease. It is believed that the sequence in which the drugs are administered is important in the effectiveness of the treatment. In how many different sequences can 5 of the 11 drugs be administered?

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30. *Insect Classification* A biologist is attempting to classify 52,000 species of insects by assigning 3 initials to each species. Is it possible to classify all the species in this way? If not, how many initials should be used?

31. *Genetics Experiment* In how many ways can 7 of 10 monkeys be arranged in a row for a genetics experiment?

SOCIAL SCIENCES

32. *Social Science Experiment* In an experiment on social interaction, 6 people will sit in 6 seats in a row. In how many ways can this be done?

33. *Election Ballots* In an election with 3 candidates for one office and 5 candidates for another office, how many different ballots may be printed?

GENERAL INTEREST

34. *Course Scheduling* A business school gives courses in typing, shorthand, transcription, business English, technical writing, and accounting. In how many ways can a student arrange a schedule if 3 courses are taken? Assume that the order in which courses are scheduled matters.

35. *Course Scheduling* If your college offers 400 courses, 20 of which are in mathematics, and your counselor arranges your schedule of 4 courses by random selection, how many schedules are possible that do not include a math course? Assume that the order in which courses are scheduled matters.

36. *Baseball Teams* A baseball team has 20 players. How many 9-player batting orders are possible?

37. *Union Elections* A chapter of union Local 715 has 35 members. In how many different ways can the chapter select a president, a vice-president, a treasurer, and a secretary?

38. *Programming Music* A concert to raise money for an economics prize is to consist of 5 works: 2 overtures, 2 sonatas, and a piano concerto.

a. In how many ways can the program be arranged?

b. In how many ways can the program be arranged if an overture must come first?

39. *Programming Music* A zydeco band from Louisiana will play 5 traditional and 3 original Cajun compositions at a concert. In how many ways can they arrange the program if

a. they begin with a traditional piece?

b. an original piece will be played last?

40. *Television Scheduling* The television schedule for a certain evening shows 8 choices from 8 to 9 P.M., 5 choices from 9 to 10 P.M., and 6 choices from 10 to 11 P.M. In how many different ways could a person schedule that evening of television viewing from 8 to 11 P.M.? (Assume each program that is selected is watched for an entire hour.)

41. *Radio Station Call Letters* How many different 4-letter radio station call letters can be made if

a. the first letter must be K or W and no letter may be repeated?

b. repeats are allowed, but the first letter is K or W?

c. the first letter is K or W, there are no repeats, and the last letter is R?

42. *Telephone Numbers* How many 7-digit telephone numbers are possible if the first digit cannot be zero and

a. only odd digits may be used?

b. the telephone number must be a multiple of 10 (that is, it must end in zero)?

c. the telephone number must be a multiple of 100?

d. the first 3 digits are 481?

e. no repetitions are allowed?

Telephone Area Codes Several years ago, the United States began running out of telephone numbers. Telephone companies introduced new area codes as numbers were used up, and eventually almost all area codes were used up.

43. a. Until recently, all area codes had a 0 or 1 as the middle digit, and the first digit could not be 0 or 1. How many area codes are there with this arrangement? How many telephone numbers does the current 7-digit sequence permit per area code? (The 3-digit sequence that follows the area code cannot start with 0 or 1. Assume there are no other restrictions.)

b. The actual number of area codes under the previous system was 152. Explain the discrepancy between this number and your answer to part a.

44. The shortage of area codes was avoided by removing the restriction on the second digit. (This resulted in problems for some older equipment, which used the second digit to determine that a long-distance call was being made.) How many area codes are available under the new system?

45. *License Plates* For many years, the state of California used 3 letters followed by 3 digits on its automobile license plates.

a. How many different license plates are possible with this arrangement?

b. When the state ran out of new numbers, the order was reversed to 3 digits followed by 3 letters. How many new license plate numbers were then possible?

c. Several years ago, the numbers described in b were also used up. The state then issued plates with 1 letter followed by 3 digits and then 3 letters. How many new license plate numbers will this provide?

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46. ***Social Security Numbers*** A social security number has 9 digits. How many social security numbers are there? The U.S. population in 2000 was about 281 million. Is it possible for every U.S. resident to have a unique social security number? (Assume no restrictions.)

47. ***Postal Zip Codes*** The U.S. Postal Service currently uses 5-digit zip codes in most areas. How many zip codes are possible if there are no restrictions on the digits used? How many would be possible if the first number could not be 0?

48. ***Postal Zip Codes*** The U.S. Postal Service is encouraging the use of 9-digit zip codes in some areas, adding 4 digits after the usual 5-digit code. How many such zip codes are possible with no restrictions?

49. ***Games*** The game of Sets* uses a special deck of cards. Each card has either one, two, or three identical shapes, all of the same color and style. There are three possible shapes: squiggle, diamond, and oval. There are three possible colors: green, purple, and red. There are three possible styles: solid, shaded, or outline. The deck consists of all possible combinations of shape, color, style, and number of shapes. How many cards are in the deck?

50. ***Games*** In the game of Scattergories,† the players take 12 turns. In each turn, a 20-sided die is rolled; each side has a letter. The players must then fill in 12 categories (e.g., vegetable, city, etc.) with a word beginning with the letter rolled. Considering that a game consists of 12 rolls of the 20-sided die, how many possible games are there?

51. ***Games*** The game of Twenty Questions consists of asking 20 questions to determine a person, place, or thing that the other person is thinking of. The first question, which is always "Is it an animal, vegetable, or mineral?" has three possible answers. All the other questions must be answered "Yes" or "No." How many possible objects can be distinguished in this game, assuming that all 20 questions are asked? Are 20 questions enough?

52. ***Traveling Salesman*** In the famous Traveling Salesman Problem, a salesman starts in any one of a set of cities, visits every city in the set once, and returns to the starting city. He would like to complete this circuit with the shortest possible distance.

a. Suppose the salesman has 10 cities to visit. Given that it does not matter what city he starts in, how many different circuits can he take?

b. The salesman decides to check all the different paths in part a to see which is shortest, but realizes that a circuit has the same distance whichever direction it is traveled. How many different circuits must he check?

c. Suppose the salesman has 70 cities to visit. Would it be feasible to have a computer check all the different circuits? Explain your reasoning.

8.2 COMBINATIONS

THINK ABOUT IT *In how many ways can a manager select 4 employees for promotion from 12 eligible employees?*

As we shall see, permutations alone cannot be used to answer this question, but combinations will provide the answer.

In the previous section, we saw that there are 60 ways that a teacher can arrange 3 of 5 different books on his desk. That is, there are 60 permutations of 5 books taken 3 at a time. Suppose now that the teacher does not wish to arrange the books on his desk, but rather wishes to choose, without regard to order, any 3 of the 5 books for a book sale to raise money for his school. In how many ways can this be done?

At first glance, we might say 60 again, but this is incorrect. The number 60 counts all possible *arrangements* of 3 books chosen from 5. The following 6 arrangements, however, would all lead to the same set of 3 books being given to the book sale.

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mystery-biography-textbook	biography-textbook-mystery
mystery-textbook-biography	textbook-biography-mystery
biography-mystery-textbook	textbook-mystery-biography

The list shows 6 different *arrangements* of 3 books, but only one *subset* of 3 books. A subset of items listed *without regard to order* is called a **combination.** The number of combinations of 5 things taken 3 at a time is written $\binom{5}{3}$, and read "5 over 3" or "5 choose 3."* Since they are subsets, combinations are *not ordered.*

To evaluate $\binom{5}{3}$, start with the $5 \cdot 4 \cdot 3$ *permutations* of 5 things taken 3 at a time. Since combinations are not ordered, find the number of combinations by dividing the number of permutations by the number of ways each group of 3 can be ordered; that is, divide by 3!.

$$\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

There are 10 ways that the teacher can choose 3 books for the book sale.

Generalizing this discussion gives the following formula for the number of combinations of n elements taken r at a time:

$$\binom{n}{r} = \frac{P(n, r)}{r!}.$$

Another version of this formula is found as follows.

$$\begin{aligned}\binom{n}{r} &= \frac{P(n, r)}{r!} \\ &= \frac{n!}{(n-r)!} \cdot \frac{1}{r!} \\ &= \frac{n!}{(n-r)!\,r!}\end{aligned}$$

The steps above lead to the following result.

COMBINATIONS

If $\binom{n}{r}$ denotes the number of combinations of n elements taken r at a time, where $r \le n$, then

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

*Other common notations for $\binom{n}{r}$ are ${}_nC_r$, C_r^n, and $C(n, r)$.

EXAMPLE 1 *Committees*

How many committees of 3 people can be formed from a group of 8 people?

Solution

Method 1: Calculating by Hand

A committee is an unordered group, so use the combinations formula for $\binom{8}{3}$.

$$\binom{8}{3} = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

Method 2: Graphing Calculator

Graphing calculators have the capacity to compute combinations. For example, on a TI-83/84 Plus, $\binom{8}{3}$ can be calculated by inputting 8 followed by `nCr` (found in the `MATH-PRB` menu) and a 3 yielding 56, as shown in Figure 4.

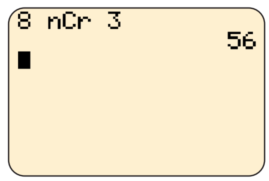

FIGURE 4

Method 3: Spreadsheet

Spreadsheets can also compute combinations. For example, in Microsoft Excel, $\binom{8}{3}$ can be calculated by inputting 8 and 3 in cells, say, A1 and B1, and then typing "`=FACT(A1)/(FACT(A1-B1)*FACT(B1))`" in cell C1 or, for that matter, any other cell.

Example 1 shows an alternative way to compute $\binom{n}{r}$. Take r or $n - r$, whichever is smaller. Write the factorial of this number in the denominator. In the numerator, write out a sufficient number of factors of $n!$ so there is one factor in the numerator for each factor in the denominator. For example, to calculate $\binom{8}{3}$ or $\binom{8}{5}$, write

$$\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56.$$

The factors that are omitted (written in color in Example 1) cancel out of the numerator and denominator, so need not be included.

Notice from the previous discussion that $\binom{8}{3} = \binom{8}{5}$. (See Exercise 25 for a generalization of this idea.) One interpretation of this fact is that the number of ways to form a committee of 3 people chosen from a group of 8 is the same as the number of ways to choose the 5 people who are not on the committee.

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EXAMPLE 2 *Lawyers*

Three lawyers are to be selected from a group of 30 to work on a special project.

(a) In how many different ways can the lawyers be selected?

Solution Here we wish to know the number of 3-element combinations that can be formed from a set of 30 elements. (We want combinations, not permutations, since order within the group of 3 doesn't matter.)

$$\binom{30}{3} = \frac{30!}{27!3!} = \frac{30 \cdot 29 \cdot 28 \cdot 27!}{27! \cdot 3 \cdot 2 \cdot 1}$$
$$= \frac{30 \cdot 29 \cdot 28}{3 \cdot 2 \cdot 1}$$
$$= 4060$$

There are 4060 ways to select the project group.

(b) In how many ways can the group of 3 be selected if a certain lawyer must work on the project?

Solution Since 1 lawyer already has been selected for the project, the problem is reduced to selecting 2 more from the remaining 29 lawyers.

$$\binom{29}{2} = \frac{29!}{27!\,2!} = \frac{29 \cdot 28 \cdot 27!}{27! \cdot 2 \cdot 1} = \frac{29 \cdot 28}{2 \cdot 1} = 29 \cdot 14 = 406$$

In this case, the project group can be selected in 406 ways.

(c) In how many ways can a nonempty group of at most 3 lawyers be selected from these 30 lawyers?

Solution Here, by "at most 3" we mean "1 or 2 or 3." (The number 0 is excluded because the group is nonempty.) Find the number of ways for each case.

Case	Number of Ways
1	$\binom{30}{1} = \frac{30!}{29!\,1!} = \frac{30 \cdot 29!}{29!\,(1)} = 30$
2	$\binom{30}{2} = \frac{30!}{28!\,2!} = \frac{30 \cdot 29 \cdot 28!}{28! \cdot 2 \cdot 1} = 435$
3	$\binom{30}{3} = \frac{30!}{27!\,3!} = \frac{30 \cdot 29 \cdot 28 \cdot 27!}{27! \cdot 3 \cdot 2 \cdot 1} = 4060$

The total number of ways to select at most 3 lawyers will be the sum

$$30 + 435 + 4060 = 4525.$$

EXAMPLE 3 *Sales*

A salesman has 10 accounts in a certain city.

(a) In how many ways can he select 3 accounts to call on?

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Solution Within a selection of 3 accounts, the arrangement of the calls is not important, so there are

$$\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

ways he can make a selection of 3 accounts.

(b) In how many ways can he select at least 8 of the 10 accounts to use in preparing a report?

Solution "At least 8" means "8 or more," which is "8 or 9 or 10." First find the number of ways to choose in each case.

Case	Number of Ways
8	$\binom{10}{8} = \frac{10!}{2!\,8!} = \frac{10 \cdot 9}{2 \cdot 1} = 45$
9	$\binom{10}{9} = \frac{10!}{1!\,9!} = \frac{10}{1} = 10$
10	$\binom{10}{10} = \frac{10!}{0!\,10!} = 1$

He can select at least 8 of the 10 accounts in $45 + 10 + 1 = 56$ ways.

FOR REVIEW

Notice in Example 3 that to calculate the number of ways to select 8 or 9 or 10 accounts, we added the three numbers found. The union rule for disjoint sets from Chapter 7 says that when A and B are disjoint sets, the number of elements in A or B is the number of elements in A plus the number in B.

CAUTION When we are making choice 1 *and* choice 2, we *multiply* to find the total number of ways. When we are making choice 1 *or* choice 2, we *add* to find the total number of ways.

The formulas for permutations and combinations given in this section and in the previous section will be very useful in solving probability problems in the next section. Any difficulty in using these formulas usually comes from being unable to differentiate between them. Both permutations and combinations give the number of ways to choose r objects from a set of n objects. The differences between permutations and combinations are outlined in the following table.

Permutations	Combinations
Different orderings or arrangements of the r objects are different permutations.	Each choice or subset of r objects gives one combination. Order within the group of r objects does not matter.
$P(n,r) = \frac{n!}{(n-r)!}$	$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$
Clue words: arrangement, schedule, order	Clue words: group, committee, set, sample
Order matters!	**Order does not matter!**

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In the next examples, concentrate on recognizing which formula should be applied.

EXAMPLE 4 *Permutations and Combinations*

For each problem, tell whether permutations or combinations should be used to solve the problem.

(a) How many 4-digit code numbers are possible if no digits are repeated?

Solution Since changing the order of the 4 digits results in a different code, use permutations.

(b) A sample of 3 light bulbs is randomly selected from a batch of 15. How many different samples are possible?

Solution The order in which the 3 light bulbs are selected is not important. The sample is unchanged if the items are rearranged, so combinations should be used.

(c) In a baseball conference with 8 teams, how many games must be played so that each team plays every other team exactly once?

Solution Selection of 2 teams for a game is an *unordered* subset of 2 from the set of 8 teams. Use combinations again.

(d) In how many ways can 4 patients be assigned to 6 different hospital rooms so that each patient has a private room?

Solution The room assignments are an *ordered* selection of 4 rooms from the 6 rooms. Exchanging the rooms of any 2 patients within a selection of 4 rooms gives a different assignment, so permutations should be used.

(e) Solve the problems in parts (a)–(d) above. The answers are given in the footnote.*

EXAMPLE 5 *Promotions*

A manager must select 4 employees for promotion; 12 employees are eligible.

(a) In how many ways can the 4 be chosen?

Solution Since there is no reason to differentiate among the 4 who are selected, use combinations.

$$\binom{12}{4} = \frac{12!}{8!\,4!} = 495$$

(b) In how many ways can 4 employees be chosen (from 12) to be placed in 4 different jobs?

Solution In this case, once a group of 4 is selected, they can be assigned in many different ways (or arrangements) to the 4 jobs. Therefore, this problem requires permutations.

$$P(12, 4) = \frac{12!}{8!} = 11{,}880$$

EXAMPLE 6 *Playing Cards*

In how many ways can a full house of aces and eights (3 aces and 2 eights) occur in 5-card poker?

*(**a**) 5040 (**b**)455 (**c**) 28 (**d**) 360

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FOR REVIEW

Examples 6 and 7 involve a standard deck of 52 playing cards, as shown in Figure 17 in Chapter 7. Recall the discussion that accompanies the photograph.

Solution The arrangement of the 3 aces or the 2 eights does not matter, so we use combinations and the multiplication principle. There are $\binom{4}{3}$ ways to get 3 aces from the 4 aces in the deck, and $\binom{4}{2}$ ways to get 2 eights. By the multiplication principle, the number of ways to get 3 aces and 2 eights is

$$\binom{4}{3} \cdot \binom{4}{2} = 4 \cdot 6 = 24.$$

EXAMPLE 7 *Playing Cards*

Five cards are dealt from a standard 52-card deck.

(a) How many such hands have only face cards?

Solution The face cards are the king, queen, and jack of each suit. Since there are 4 suits, there are 12 face cards. The arrangement of the 5 cards is not important, so use combinations to get

$$\binom{12}{5} = \frac{12!}{7!\,5!} = 792.$$

(b) How many such hands have exactly 2 hearts?

Solution There are 13 hearts in the deck, so the 2 hearts will be selected from those 13 cards. The other 3 cards must come from the remaining 39 cards that are not hearts. Use combinations and the multiplication principle to get

$$\binom{13}{2}\binom{39}{3} = 78 \cdot 9139 = 712{,}842.$$

Notice that the two top numbers in the combinations add up to 52, the total number of cards, and the two bottom numbers add up to 5, the number of cards in a hand.

(c) How many such hands have cards of a single suit?

Solution The total number of ways that 5 cards of a particular suit of 13 cards can occur is $\binom{13}{5}$. Since the arrangement of the 5 cards is not important, use combinations. There are four different suits, so the multiplication principle gives

$$4 \cdot \binom{13}{5} = 4 \cdot 1287 = 5148$$

ways to deal 5 cards of the same suit.

As Example 7 shows, often both combinations and the multiplication principle must be used in the same problem.

EXAMPLE 8 *Soup*

To illustrate the differences between permutations and combinations in another way, suppose 2 cans of soup are to be selected from 4 cans on a shelf: noodle (N), bean (B), mushroom (M), and tomato (T). As shown in Figure 5(a) on the next page, there are 12 ways to select 2 cans from the 4 cans if the order matters (if

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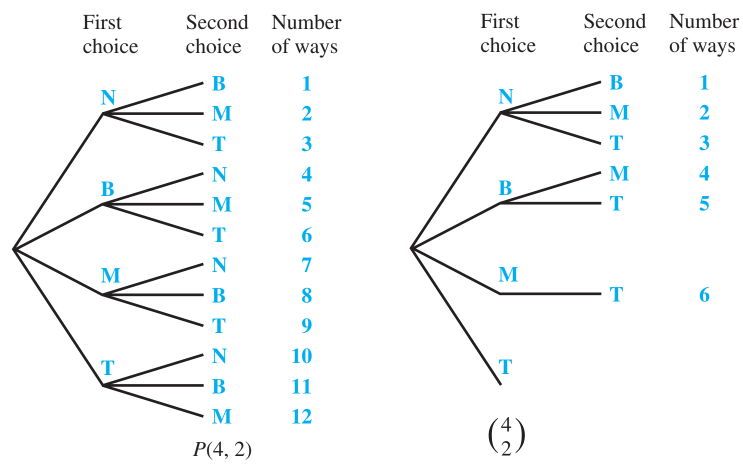
noodle first and bean second is considered different from bean, then noodle, for example). On the other hand, if order is unimportant, then there are 6 ways to choose 2 cans of soup from the 4, as illustrated in Figure 5(b).



(a) **(b)**

FIGURE 5

CAUTION It should be stressed that not all counting problems lend themselves to either permutations or combinations. Whenever a tree diagram or the multiplication principle can be used directly, it's often best to use it. ■

8.2 EXERCISES

1. Define combinations in your own words.

Evaluate each combination.

2. $\binom{8}{3}$ **3.** $\binom{12}{5}$ **4.** $\binom{44}{20}$ **5.** $\binom{40}{18}$

6. $\binom{n}{0}$ **7.** $\binom{n}{n}$ **8.** $\binom{n}{1}$ **9.** $\binom{n}{n-1}$

10. In how many ways can a hand of 6 clubs be chosen from an ordinary deck?

11. Five cards are marked with the numbers 1, 2, 3, 4, and 5, then shuffled, and 2 cards are drawn.

a. How many different 2-card combinations are possible?

b. How many 2-card hands contain a number less than 3?

12. An economics club has 30 members.

a. If a committee of 4 is to be selected, in how many ways can the selection be made?

b. In how many ways can a committee of at least 1 and at most 3 be selected?

13. Use a tree diagram for the following.

a. Find the number of ways 2 letters can be chosen from the set $\{L, M, N\}$ if order is important and repetition is allowed.

b. Reconsider part a if no repeats are allowed.

c. Find the number of combinations of 3 elements taken 2 at a time. Does this answer differ from part a or b?

14. Repeat Exercise 13 using the set $\{L, M, N, P\}$.

15. Explain the difference between a permutation and a combination.

16. Padlocks with digit dials are often referred to as "combination locks." According to the mathematical definition of combination, is this an accurate description? Explain.

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Decide whether each exercise involves permutations or combinations, and then solve the problem.

17. In a club with 8 male and 11 female members, how many 5-member committees can be chosen that have

a. all men? **b.** all women?

c. 3 men and 2 women?

18. In Exercise 17, how many committees can be selected that have

a. at least 4 women? **b.** no more than 2 men?

19. In a game of musical chairs, 12 children will sit in 11 chairs arranged in a row (one will be left out). In how many ways can this happen, if we count rearrangements of the children in the chairs as different outcomes?

20. A group of 3 students is to be selected from a group of 12 students to take part in a class in cell biology.

a. In how many ways can this be done?

b. In how many ways can the group who will not take part be chosen?

21. Marbles are being drawn without replacement from a bag containing 15 marbles.

a. How many samples of 2 marbles can be drawn?

b. How many samples of 4 marbles can be drawn?

c. If the bag contains 3 yellow, 4 white, and 8 blue marbles, how many samples of 2 marbles can be drawn in which both marbles are blue?

22. There are 5 rotten apples in a crate of 25 apples.

a. How many samples of 3 apples can be drawn from the crate?

b. How many samples of 3 could be drawn in which all 3 are rotten?

c. How many samples of 3 could be drawn in which there are two good apples and one rotten one?

23. A bag contains 5 black, 1 red, and 3 yellow jelly beans; you take 3 at random. How many samples are possible in which the jelly beans are

a. all black? **b.** all red?

c. all yellow? **d.** 2 black and 1 red?

e. 2 black and 1 yellow? **f.** 2 yellow and 1 black?

g. 2 red and 1 yellow?

24. In how many ways can 5 out of 9 plants be arranged in a row on a windowsill?

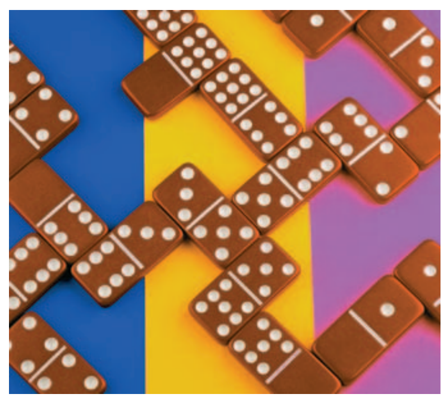
25. Show that $\binom{n}{r} = \binom{n}{n-r}$.

26. The following problem was posed on National Public Radio's *Weekend Edition:* In how many points can 6 circles intersect?*

a. Find the answer for 6 circles.

b. Find the general answer for n circles.

27. How many different dominoes can be formed from the numbers $0 \ldots 6$? (*Hint:* A domino may have the same number of dots on both halves of it or it may have a different number of dots on each half.)



Applications

BUSINESS AND ECONOMICS

28. ***Secretarial Assignments*** From a pool of 7 secretaries, 3 are selected to be assigned to 3 managers, one per manager. In how many ways can they be selected and assigned?

29. ***Sales Schedules*** A salesperson has the names of 6 prospects.

a. In how many ways can she arrange her schedule if she calls on all 6?

b. In how many ways can she arrange her schedule if she can call on only 4 of the 6?

30. ***Worker Grievances*** A group of 7 workers decides to send a delegation of 2 to their supervisor to discuss their grievances.

a. How many delegations are possible?

b. If it is decided that a particular worker must be in the delegation, how many different delegations are possible?

**Weekend Edition,* National Public Radio, Oct. 23, 1994.

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c. If there are 2 women and 5 men in the group, how many delegations would include at least 1 woman?

31. ***Hamburger Variety*** Hamburger Hut sells regular hamburgers as well as a larger burger. Either type can include cheese, relish, lettuce, tomato, mustard, or catsup.

a. How many different hamburgers can be ordered with exactly three extras?

b. How many different regular hamburgers can be ordered with exactly three extras?

c. How many different regular hamburgers can be ordered with at least five extras?

32. ***Assembly Line Sampling*** Five items are to be randomly selected from the first 50 items on an assembly line to determine the defect rate. How many different samples of 5 items can be chosen?

LIFE SCIENCES

33. ***Research Participants*** From a group of 16 smokers and 20 nonsmokers, a researcher wants to randomly select 8 smokers and 8 nonsmokers for a study. In how many ways can the study group be selected?

34. ***Plant Hardiness*** In an experiment on plant hardiness, a researcher gathers 6 wheat plants, 3 barley plants, and 2 rye plants. She wishes to select 4 plants at random.

a. In how many ways can this be done?

b. In how many ways can this be done if exactly 2 wheat plants must be included?

SOCIAL SCIENCES

35. ***Legislative Committee*** A legislative committee consists of 5 Democrats and 4 Republicans. A delegation of 3 is to be selected to visit a small Pacific island republic.

a. How many different delegations are possible?

b. How many delegations would have all Democrats?

c. How many delegations would have 2 Democrats and 1 Republican?

d. How many delegations would include at least 1 Republican?

36. ***Political Committee*** From 10 names on a ballot, 4 will be elected to a political party committee. In how many ways can the committee of 4 be formed if each person will have a different responsibility, and different assignments of responsibility are considered different committees?

37. ***Judges*** When Paul Martinek, publisher of *Lawyers Weekly USA,* was a guest on the television news program *The O'Reilly Factor,* he discussed a decision by a three-judge panel, chosen at random from judges in the Ninth Circuit in California.* The judges had ruled that the mandatory recitation of the Pledge of Allegience is unconstitutional because of the phrase "under God." According to Martinek, "Because there are 45 judges in the Ninth Circuit, there are 3000 different combinations of three-judge panels." Is this true? If not, what is the correct number?

GENERAL INTEREST

38. ***Bridge*** How many different 13-card bridge hands can be selected from an ordinary deck?

39. ***Poker*** Five cards are chosen from an ordinary deck to form a hand in poker. In how many ways is it possible to get the following results?

a. 4 queens

b. No face card

c. Exactly 2 face cards

d. At least 2 face cards

e. 1 heart, 2 diamonds, and 2 clubs

40. ***Baseball*** If a baseball coach has 5 good hitters and 4 poor hitters on the bench and chooses 3 players at random, in how many ways can he choose at least 2 good hitters?

41. ***Softball*** The coach of the Morton Valley Softball Team has 6 good hitters and 8 poor hitters. He chooses 3 hitters at random.

a. In how many ways can he choose 2 good hitters and 1 poor hitter?

b. In how many ways can he choose 3 good hitters?

c. In how many ways can he choose at least 2 good hitters?

42. ***Flower Selection*** Five orchids from a collection of 20 are to be selected for a flower show.

a. In how many ways can this be done?

b. In how many ways can the 5 be selected if 2 special plants must be included?

43. ***Ice Cream Flavors*** Baskin-Robbins advertises that it has 31 flavors of ice cream.

a. How many different double-scoop cones can be made? Assume that the order of the scoops matters.

b. How many different triple-scoop cones can be made?

c. How many different double-scoop cones can be made if order doesn't matter?

44. ***Lottery*** A state lottery game requires that you pick 6 different numbers from 1 to 99. If you pick all 6 winning numbers, you win the jackpot.

**Mathematics Teacher,* Vol. 96, No. 3, March 2003.

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a. How many ways are there to choose 6 numbers if order is not important?

b. How many ways are there to choose 6 numbers if order matters?

45. ***Lottery*** In Exercise 44, if you pick 5 of the 6 numbers correctly, you win \$250,000. In how many ways can you pick exactly 5 of the 6 winning numbers without regard to order?

46. ***Pizza Varieties*** A television commercial for Little Caesars pizza announced that with the purchase of two pizzas, one could receive free any combination of up to five toppings on each pizza. The commercial shows a young child waiting in line at Little Caesars who calculates that there are 1,048,576 possibilities for the toppings on the two pizzas.*

a. Verify the child's calculation. Use the fact that Little Caesars has 11 toppings to choose from. Assume that the order of the two pizzas matters; that is, if the first pizza has combination 1 and the second pizza has combination 2, that is different from combination 2 on the first pizza and combination 1 on the second.

b. In a letter to *The Mathematics Teacher,* Joseph F. Heiser argued that the two combinations described in part a should be counted as the same, so the child has actually overcounted. Give the number of possibilities if the order of the two pizzas doesn't matter.

47. ***Cereal*** The Post Corporation has introduced the cereal, *Create a Crunch*™, in which the consumers can combine ingredients to create their own unique cereal. Each box contains 8 packets of food goods. There are four types of cereal: Frosted Alpha Bits®, Cocoa Pebbles®, Fruity Pebbles®, and Honey Comb®. Also included in the box are four "Add-Ins": granola, blue rice cereal, marshmallows, and sprinkles.

a. What is the total number of breakfasts that can be made if a breakfast is defined as any one or more cereals or add-ins?

b. If Emily Friedrich chooses to mix one type of cereal with one add-in, how many different breakfasts can she make?

c. If Rachel Moldovan chooses to mix two types of cereal with three add-ins, how many different breakfasts can she make?

d. If Vincent Sonoga chooses to mix at least one type of cereal with at least one type of add-in, how many breakfasts can he make?

e. If Ann Lombardi's favorite cereal is Fruity Pebbles®, how many different cereals can she make if each of her mixtures must include this cereal?

48. ***Football*** Writer Gregg Easterbrook, discussing ESPN's unsuccessful attempt to predict the winners for the six National Football League (NFL) divisions and the six wild-card slots, claimed that there were 180 different ways to make this forecast.† Reader Milton Eisner wrote in to tell him that the actual number is much larger.‡ To make the calculation, note that the NFL consists of two conferences, each of which consists of three divisions. Five of the divisions have five teams, while the other has six. There is one winner from each of the six divisions, plus three wild-card slots from each of the two conferences. How many ways can the six division winners and six wild-card slots be chosen?

8.3 PROBABILITY APPLICATIONS OF COUNTING PRINCIPLES

THINK ABOUT IT *If 3 engines are tested from a shipping container packed with 12 diesel engines, 2 of which are defective, what is the probability that at least 1 of the defective engines will be found (in which case the container will not be shipped)?*

This problem theoretically could be solved with a tree diagram, but it would require a tree with a large number of branches. Many of the probability problems involving *dependent* events that were solved earlier by using tree diagrams can also be solved by using permutations or combinations. Permutations and combinations are especially helpful when the numbers involved are large.

*Heiser, Joseph F., "Pascal and Gauss Meet Little Caesars," *Mathematics Teacher,* Vol. 87, Sept. 1994, p. 389.
†http://slate.msn.com/id/95622/.
‡http://slate.msn.com/id/96439/.

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To compare the method of using permutations or combinations with the method of tree diagrams used in Section 7.5, the first example repeats Example 6 from that section.

EXAMPLE 1 *Marbles*

From a box containing 3 white, 2 green, and 1 red marble, 2 marbles are drawn one at a time without replacement. Find the probability that 1 white and 1 green marble are drawn.

FOR REVIEW

The use of combinations to solve probability problems depends on the basic probability principle introduced earlier and repeated here:
Let S be a sample space with equally likely outcomes, and let event E be a subset of S. Then the probability that event E occurs, written $P(E)$, is

$$P(E) = \frac{n(E)}{n(S)},$$

where $n(E)$ and $n(S)$ represent the number of elements in sets E and S.

Solution Because the marbles are drawn one at a time, with one labeled as the first marble and the other as the second, we use permutations. There are two ways to draw 1 white marble and 1 green marble. The first way is to draw the white marble followed by the green, and the second is to draw the green followed by the white. The white marble can be drawn from the 3 white marbles in $\binom{3}{1}$ ways, and the green can be drawn from the 2 green marbles in $\binom{2}{1}$ ways. By the multiplication principle and the union rule for disjoint sets, both results can occur in

$$\binom{3}{1}\binom{2}{1} + \binom{2}{1}\binom{3}{1} \text{ ways,}$$

giving the numerator of the probability fraction, $P(E) = m/n$. For the denominator, there are 6 ways to draw the first marble and 5 ways to draw the second, for a total of $6 \cdot 5$ ways. The required probability is

$$\begin{aligned} P(1 \text{ white and } 1 \text{ green}) &= \frac{\binom{3}{1}\binom{2}{1} + \binom{2}{1}\binom{3}{1}}{6 \cdot 5} \\ &= \frac{3 \cdot 2 + 2 \cdot 3}{30} = \frac{12}{30} = \frac{2}{5}. \end{aligned}$$

This agrees with the answer found earlier.

This example can be solved more simply by observing that the probability that 1 white marble and 1 green marble are drawn should not depend upon the order in which the marbles are drawn, so we may use combinations. The numerator is simply the number of ways of drawing a white marble out of 3 white marbles and a green marble out of 2 green marbles. The denominator is just the number of ways of drawing 2 marbles out of 6. Then

$$P(1 \text{ white and } 1 \text{ green}) = \frac{\binom{3}{1}\binom{2}{1}}{\binom{6}{2}} = \frac{6}{15} = \frac{2}{5}.$$

This helps explain why combinations tend to be used more often than permutations in probability. Even if order matters in the original problem, it is sometimes possible to ignore order and use combinations. Be careful to do this only when the final result does not depend on the order of events. Order often does matter. (If you don't believe that, try getting dressed tomorrow morning and then taking your shower.)

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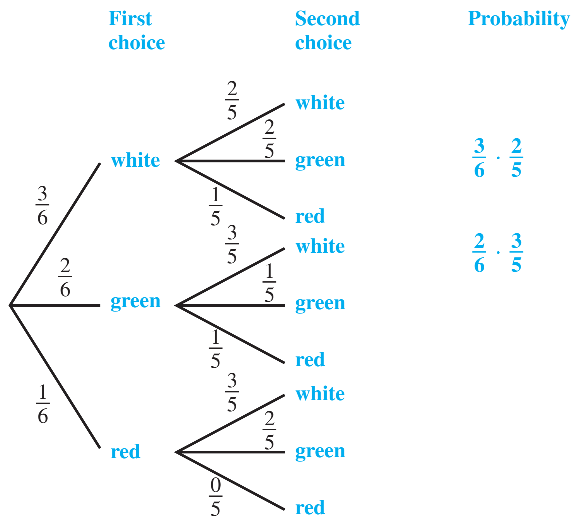

FIGURE 6

Example 1 could also be solved using the tree diagram shown in Figure 6. Two of the branches correspond to drawing 1 white and 1 green marble. The probability for each branch is calculated by multiplying the probabilities along the branch, as we did in the previous chapter. The resulting probabilities for the two branches are then added, giving the result

$$P(1 \text{ white and } 1 \text{ green}) = \frac{3}{6} \cdot \frac{2}{5} + \frac{2}{6} \cdot \frac{3}{5} = \frac{2}{5}.$$

EXAMPLE 2 *Nursing*

From a group of 22 nurses, 4 are to be selected to present a list of grievances to management.

(a) In how many ways can this be done?

Solution Four nurses from a group of 22 can be selected in $\binom{22}{4}$ ways. (Use combinations, since the group of 4 is an unordered set.)

$$\binom{22}{4} = \frac{22!}{18!\,4!} = \frac{(22)(21)(20)(19)}{(4)(3)(2)(1)} = 7315$$

There are 7315 ways to choose 4 people from 22.

(b) One of the nurses is Julie Davis. Find the probability that Davis will be among the 4 selected.

Solution The probability that Davis will be selected is given by m/n, where m is the number of ways the chosen group includes her, and n is the total number of ways the group of 4 can be chosen. If Davis must be one of the 4 selected, the problem reduces to finding the number of ways that the 3 additional nurses can be chosen. The 3 are chosen from 21 nurses; this can be done in

$$\binom{21}{3} = \frac{21!}{18!\,3!} = 1330$$

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ways, so $m = 1330$. Since n is the number of ways 4 nurses can be selected from 22,

$$n = \binom{22}{4} = 7315.$$

The probability that Davis will be one of the 4 chosen is

$$P(\text{Davis is chosen}) = \frac{1330}{7315} \approx .182.$$

(c) Find the probability that Davis will not be selected.

Solution The probability that she will not be chosen is $1 - .182 = .818$.

EXAMPLE 3 *Diesel Engines*

When shipping diesel engines abroad, it is common to pack 12 engines in one container that is then loaded on a rail car and sent to a port. Suppose that a company has received complaints from its customers that many of the engines arrive in nonworking condition. To help solve this problem, the company decides to make a spot check of containers after loading. The company will test 3 engines from a container at random; if any of the 3 are nonworking, the container will not be shipped until each engine in it is checked. Suppose a given container has 2 nonworking engines. Find the probability that the container will not be shipped.

Solution The container will not be shipped if the sample of 3 engines contains 1 or 2 defective engines. If $P(1 \text{ defective})$ represents the probability of exactly 1 defective engine in the sample, then

$$P(\text{not shipping}) = P(1 \text{ defective}) + P(2 \text{ defective}).$$

There are $\binom{12}{3}$ ways to choose the 3 engines for testing:

$$\binom{12}{3} = \frac{12!}{9!\,3!} = 220.$$

There are $\binom{2}{1}$ ways of choosing 1 defective engine from the 2 in the container, and for each of these ways, there are $\binom{10}{2}$ ways of choosing 2 good engines from among the 10 in the container. By the multiplication principle, there are

$$\binom{2}{1}\binom{10}{2} = \frac{2!}{1!1!} \cdot \frac{10!}{8!2!} = 90$$

ways of choosing a sample of 3 engines containing 1 defective engine with

$$P(1 \text{ defective}) = \frac{90}{220} = \frac{9}{22}.$$

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There are $\binom{2}{2}$ ways of choosing 2 defective engines from the 2 defective engines in the container, and $\binom{10}{1}$ ways of choosing 1 good engine from among the 10 good engines, for

$$\binom{2}{2}\binom{10}{1} = \frac{2!}{0!\,2!} \cdot \frac{10!}{9!\,1!} = 10$$

ways of choosing a sample of 3 engines containing 2 defective engines. Finally,

$$P(2 \text{ defective}) = \frac{10}{220} = \frac{1}{22}$$

and

$$\begin{aligned} P(\text{not shipping}) &= P(1 \text{ defective}) + P(2 \text{ defective}) \\ &= \frac{9}{22} + \frac{1}{22} = \frac{10}{22} \approx .455. \end{aligned}$$

Notice that the probability is $1 - .455 = .545$ that the container will be shipped, even though it has 2 defective engines. The management must decide whether this probability is acceptable; if not, it may be necessary to test more than 3 engines from a container.

FOR REVIEW

Recall that if E and E' are complements, then $P(E') = 1 - P(E)$. In Example 3, the event "0 defective in the sample" is the complement of the event "1 or 2 defective in the sample," since there are only 0 or 1 or 2 defective engines possible in the sample of 3 engines.

Instead of finding the sum $P(1 \text{ defective}) + P(2 \text{ defective})$, the result in Example 3 could be found as $1 - P(0 \text{ defective})$.

$$\begin{aligned} P(\text{not shipping}) &= 1 - P(0 \text{ defective in sample}) \\ &= 1 - \frac{\binom{2}{0}\binom{10}{3}}{\binom{12}{3}} \\ &= 1 - \frac{1(120)}{220} \\ &= 1 - \frac{120}{220} = \frac{100}{220} \approx .455 \end{aligned}$$

EXAMPLE 4 *Poker*

In a common form of the card game *poker,* a hand of 5 cards is dealt to each player from a deck of 52 cards. There are a total of

$$\binom{52}{5} = \frac{52!}{47!\,5!} = 2{,}598{,}960$$

such hands possible. Find the probability of getting each hand.

(a) A hand containing only hearts, called a *heart flush*

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Solution There are 13 hearts in a deck, with

$$\binom{13}{5} = \frac{13!}{8!\,5!} = \frac{(13)(12)(11)(10)(9)}{(5)(4)(3)(2)(1)} = 1287$$

different hands containing only hearts. The probability of a heart flush is

$$P(\text{heart flush}) = \frac{1287}{2{,}598{,}960} \approx .000495.$$

(b) A flush of any suit (5 cards of the same suit)

Solution There are 4 suits in a deck, so

$$P(\text{flush}) = 4 \cdot P(\text{heart flush}) = 4 \cdot .000495 \approx .00198.$$

(c) A full house of aces and eights (3 aces and 2 eights)

Solution There are $\binom{4}{3}$ ways to choose 3 aces from among the 4 in the deck, and $\binom{4}{2}$ ways to choose 2 eights.

$$P(3 \text{ aces, } 2 \text{ eights}) = \frac{\binom{4}{3} \cdot \binom{4}{2}}{2{,}598{,}960} = \frac{4 \cdot 6}{2{,}598{,}960} \approx .00000923$$

(d) Any full house (3 cards of one value, 2 of another)

Solution The 13 values in a deck give 13 choices for the first value. As in part (c), there are $\binom{4}{3}$ ways to choose the 3 cards from among the 4 cards that have that value. This leaves 12 choices for the second value (order *is* important here, since a full house of 3 aces and 2 eights is not the same as a full house of 3 eights and 2 aces). From the 4 cards that have the second value, there are $\binom{4}{2}$ ways to choose 2. The probability of any full house is then

$$P(\text{full house}) = \frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}}{2{,}598{,}960} \approx .00144.$$

EXAMPLE 5 *Music*

A music teacher has 3 violin pupils, Fred, Carl, and Helen. For a recital, the teacher selects a first violinist and a second violinist. The third pupil will play with the others, but not solo. If the teacher selects randomly, what is the probability that Helen is first violinist, Carl is second violinist, and Fred does not solo?

Solution Use *permutations* to find the number of arrangements in the sample space.

$$P(3, 3) = 3! = 6$$

(Think of this as filling the positions of first violin, second violin, and no solo.) The 6 arrangements are equally likely, since the teacher will select randomly. Thus, the required probability is $1/6$.

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EXAMPLE 6 *Birthdays*

Suppose a group of n people is in a room. Find the probability that at least 2 of the people have the same birthday.

Solution "Same birthday" refers to the month and the day, not necessarily the same year. Also, ignore leap years, and assume that each day in the year is equally likely as a birthday. To see how to proceed, we first find the probability that *no 2 people* from among 5 people have the same birthday. There are 365 different birthdays possible for the first of the 5 people, 364 for the second (so that the people have different birthdays), 363 for the third, and so on. The number of ways the 5 people can have different birthdays is thus the number of permutations of 365 days taken 5 at a time or

$$P(365, 5) = 365 \cdot 364 \cdot 363 \cdot 362 \cdot 361.$$

The number of ways that 5 people can have the same birthday or different birthdays is

$$365 \cdot 365 \cdot 365 \cdot 365 \cdot 365 = (365)^5.$$

Finally, the *probability* that none of the 5 people have the same birthday is

$$\frac{P(365, 5)}{(365)^5} = \frac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361}{365 \cdot 365 \cdot 365 \cdot 365 \cdot 365} \approx .973.$$

The probability that at least 2 of the 5 people *do* have the same birthday is $1 - .973 = .027$.

Now this result can be extended to more than 5 people. Generalizing, the probability that no 2 people among n people have the same birthday is

$$\frac{P(365, n)}{(365)^n}.$$

The probability that at least 2 of the n people *do* have the same birthday is

$$1 - \frac{P(365, n)}{(365)^n}.$$

The following table shows this probability for various values of n.

Number of People, n	Probability That Two Have the Same Birthday
5	.027
10	.117
15	.253
20	.411
22	.476
23	.507
25	.569
30	.706
35	.814
40	.891
50	.970
366	1

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The probability that 2 people among 23 have the same birthday is .507, a little more than half. Many people are surprised at this result; it seems that a larger number of people should be required.

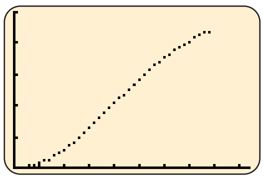

FIGURE 7

Using a graphing calculator, we can graph the probability formula in the previous example as a function of n, but care must be taken that the graphing calculator evaluates the function at integer points. Figure 7 was produced on a TI-83/84 Plus by letting `Y1 = 1 − (365 nPr X)/365^X` on $0 \le x \le 47$. (This domain ensures integer values for x.) Notice that the graph does not extend past $x = 39$. This is because $P(365, n)$ and 365^n are too large for the calculator when $n \ge 40$.

An alternative way of doing the calculations that does not run into such large numbers is based on the concept of conditional probability. The probability that the first person's birthday does not match any so far is $365/365$. The probability that the second person's birthday does not match the first's is $364/365$. The probability that the third person's birthday does not match the first's or the second's is $363/365$. By the product rule of probability, the probability that none of the first 3 people have matching birthdays is

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365}.$$

Similarly, the probability that no two people in a group of 40 have the same birthday is

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \cdots \cdot \frac{326}{365}.$$

This probability can be calculated (and then subtracted from 1 to get the probability we seek) without overflowing the calculator by multiplying each fraction times the next, rather then trying to compute the entire numerator and the entire denominator. The calculations are somewhat tedious to do by hand, but can be programmed on a graphing calculator or computer.

8.3 EXERCISES

A basket contains 6 red apples and 4 yellow apples. A sample of 3 apples is drawn. Find the probabilities that the sample contains the following.

1. All red apples

2. All yellow apples

3. 2 yellow and 1 red apple

4. More red than yellow apples

Two cards are drawn at random from an ordinary deck of 52 cards.

5. How many 2-card hands are possible?

Find the probability that the 2-card hand described above contains the following.

6. 2 aces

7. At least 1 ace

8. All spades

9. 2 cards of the same suit

10. Only face cards

11. No face cards

12. No card higher than 8 (count ace as 1)

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Twenty-six slips of paper are each marked with a different letter of the alphabet and placed in a basket. A slip is pulled out, its letter recorded (in the order in which the slip was drawn), and the slip is replaced. This is done 5 times. Find the probabilities that the following "words" are formed.

13. Chuck

14. A word that starts with p

15. A word with no repetition of letters

16. A word that contains no x, y, or z

17. Discuss the relative merits of using tree diagrams versus combinations to solve probability problems. When would each approach be most appropriate?

18. Several examples in this section used the rule $P(E') = 1 - P(E)$. Explain the advantage (especially in Example 6) of using this rule.

For Exercises 19–22, refer to Example 6 in this section.

19. A total of 42 men have served as president through 2004.* Set up the probability that, if 42 men were selected at random, at least 2 have the same birthday.†

20. Set up the probability that at least 2 of the 100 U.S. senators have the same birthday.

21. What is the probability that at least 2 of the 435 members of the House of Representatives have the same birthday?

22. Argue that the probability that in a group of n people *exactly one pair* have the same birthday is

$$\binom{n}{2} \cdot \frac{P(365, n - 1)}{(365)^n}.$$

23. An elevator has 4 passengers and stops at 7 floors. It is equally likely that a person will get off at any one of the 7 floors. Find the probability that no 2 passengers leave at the same floor.

24. On National Public Radio, the *Weekend Edition* program on Sunday, September 7, 1991, posed the following probability problem: Given a certain number of balls, of which some are blue, pick 5 at random. The probability that all 5 are blue is 1/2. Determine the original number of balls and decide how many were blue.

25. A reader wrote to the "Ask Marilyn" column‡ in *Parade* magazine, "You have six envelopes to pick from. Two-thirds (that is, four) are empty. One-third (that is, two) contain a $100 bill. You're told to choose 2 envelopes at random. Which is more likely: (1) that you'll get at least one $100 bill, or (2) that you'll get no $100 bill at all?" Find the two probabilities.

26. After studying all night for a final exam, a bleary-eyed student randomly grabs 2 socks from a drawer containing 9 black, 6 brown, and 2 blue socks, all mixed together. What is the probability that she grabs a matched pair?

27. Three crows, 4 blue jays, and 5 starlings sit in a random order on a section of telephone wire. Find the probability that birds of a feather flock together, that is, that all birds of the same type are sitting together.

28. If the letters l, i, t, t, l, and e are chosen at random, what is the probability that they spell the word "little"?

*Although Bush is the 43rd president, the 22nd and 24th presidents were the same man: Grover Cleveland.

†In fact, James Polk and Warren Harding were both born on November 2.

‡*Parade* magazine, Apr. 30, 1995, p. 8. Reprinted by permission of the William Morris Agency, Inc. on behalf of the author. Copyright © 1995 by Marilyn vos Savant.

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29. If the letters M, i, s, s, i, s, s, i, p, p, and i are chosen at random, what is the probability that they spell the word "Mississippi"?

Applications

BUSINESS AND ECONOMICS

Quality Control *A shipment of* 9 *typewriters contains* 2 *that are defective. Find the probability that a sample of the following sizes, drawn from the* 9, *will not contain a defective typewriter.*

30. 1 **31.** 2 **32.** 3 **33.** 4

Refer to Example 3. The managers feel that the probability of .545 that a container will be shipped even though it contains 2 *defective engines is too high. They decide to increase the sample size chosen. Find the probabilities that a container will be shipped even though it contains* 2 *defective engines, if the sample size is increased to the following.*

34. 4 **35.** 5

SOCIAL SCIENCES

36. ***Election Ballots*** Five names are put on a ballot in a randomly selected order. What is the probability that they are not in alphabetical order?

37. ***Native American Council*** At the first meeting of a committee to plan a Northern California pow-wow, there were 3 women and 3 men from the Miwok tribe, 2 men and 3 women from the Hoopa tribe, and 4 women and 5 men from the Pomo tribe. If the ceremony subcouncil consists of 5 people, and is randomly selected, find the probabilities that the subcouncil contains

a. 3 men and 2 women;

b. exactly 3 Miwoks and 2 Pomos;

c. 2 Miwoks, 2 Hoopas, and a Pomo;

d. 2 Miwoks, 2 Hoopas, and 2 Pomos;

e. more women than men;

f. exactly 3 Hoopas;

g. at least 2 Pomos.

38. ***Education*** A school in Bangkok requires that students take an entrance examination. After the examination, there is a drawing where 5 students are randomly selected from each group of 40 for automatic acceptance into the school, regardless of their performance on the examination. The drawing consists of placing 35 red and 5 green pieces of paper into a box. Each student picks a piece of paper from the box and then does not return the piece of paper to the box. The 5 lucky students who pick the green pieces are automatically accepted into the school.*

a. What is the probability that the first person wins automatic acceptance?

b. What is the probability that the last person wins automatic acceptance?

c. If the students are chosen by the order of their seating, does this give the student who goes first a better chance of winning than the second, third, . . . person? (*Hint:* Imagine that the 40 pieces of paper have been mixed up and laid in a row so that the first student picks the first piece of paper, the second student picks the second piece of paper, and so on.)

GENERAL INTEREST

Poker *Find the probabilities of the following hands at poker. Assume aces are either high or low.*

39. Royal flush (5 highest cards of a single suit)

40. Straight flush (5 in a row in a single suit, but not a royal flush)

41. Four of a kind (4 cards of the same value)

42. Straight (5 cards in a row, not all of the same suit), with ace either high or low

43. Three of a kind (3 cards of one value, with the other cards of two different values)

44. Two pairs (2 cards of one value, 2 of another value, and 1 of a third value)

45. One pair (2 cards of one value, with the other cards of three different values)

Bridge *A bridge hand is made up of* 13 *cards from a deck of* 52. *Find the probabilities that a hand chosen at random contains the following.*

46. Only hearts

47. 4 aces

48. Exactly 3 aces and exactly 3 kings

*Letter to the editor, *Mathematics Teacher,* Vol. 92, No. 8, Nov. 1999.

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49. 6 of one suit, 5 of another, and 2 of another

50. ***Writers*** At a conference of African American writers in Detroit, special-edition books were selected to be given away in contests. There were 9 books written by Langston Hughes, 5 books by James Baldwin, and 7 books by Toni Morrison. The judge of one contest selected 6 books at random for prizes. Find the probabilities that the selection consisted of the following.

a. 3 Hughes and 3 Morrison books

b. Exactly 4 Baldwin books

c. 2 Hughes, 3 Baldwin, and 1 Morrison book

d. At least 4 Hughes books

e. Exactly 4 books written by males (Morrison is female)

f. No more than 2 books written by Baldwin

51. ***Lottery*** In the previous section, we found the number of ways to pick 6 different numbers from 1 to 99 in a state lottery. Assuming order is unimportant, what is the probability of picking all 6 numbers correctly to win the big prize?

52. ***Lottery*** In Exercise 51, what is the probability of picking exactly 5 of the 6 numbers correctly?

53. ***Lottery*** An article in *The New York Times* discussing the odds of winning the lottery stated, "And who cares if a game-theory professor once calculated the odds of winning as equal to a poker player's chance of drawing four royal flushes in a row, all in spades—then getting up from the card table and meeting four strangers, all with the same birthday?"* Calculate this probability. Does this probability seem comparable to the odds of winning the lottery? (Ignore February 29 as a birthday, and assume that all four strangers have the same birthday as each other, not necessarily the same as the poker player.)

54. ***Barbie*** A controversy arose in 1992 over the Teen Talk Barbie doll, each of which was programmed with four sayings randomly picked from a set of 270 sayings. The controversy was over the saying, "Math class is tough," which some felt gave a negative message toward girls doing well in math. In an interview with *Science*, a spokeswoman for Mattel, the makers of Barbie, said that "There's a less than 1% chance you're going to get a doll that says math class is tough."† Is this figure correct? If not, give the correct figure.

55. ***Football*** During the 1988 college football season, the Big Eight Conference ended the season in a "perfect progression," as shown in the following table.‡

Won	Lost	Team
7	0	Nebraska (NU)
6	1	Oklahoma (OU)
5	2	Oklahoma State (OSU)
4	3	Colorado (CU)
3	4	Iowa State (ISU)
2	5	Missouri (MU)
1	6	Kansas (KU)
0	7	Kansas State (KSU)

Someone wondered what the probability of such an outcome might be.

a. How many games do the 8 teams play?

b. Assuming no ties, how many different outcomes are there for all the games together?

c. In how many ways could the 8 teams end in a perfect progression?

d. Assuming that each team had an equally likely probability of winning each game, find the probability of a perfect progression with 8 teams.

e. Find a general expression for the probability of a perfect progression in an n-team league with the same assumptions.

56. ***Bingo*** Bingo has become popular in the United States, and it is an efficient way for many organizations to raise money. The bingo card has 5 rows and 5 columns of numbers from 1 to 75, with the center given as a free cell. Balls showing one of the 75 numbers are picked at random from a container. If the drawn number appears on a player's card, then the player covers the number. In general, the winner is the person who has a card with an entire row, column, or diagonal covered.§

a. Find the probability that a person will win bingo after just four numbers are called.

*Gould, Lois, "Ticket to Trouble," *The New York Times Magazine,* Apr. 23, 1995, p. 39.
†*Science,* Vol. 258, Oct. 16, 1992, p. 398.
‡Madsen, Richard, "On the Probability of a Perfect Progression," *The American Statistician,* Aug. 1991, Vol. 45, No. 3, p. 214.
§Bay, Jennifer M., Robert E. Reys, Ken Simms, and P. Mark Taylor, "Bingo Games: Turning Student Intuitions into Investigations in Probability and Number Sense," *Mathematics Teacher,* Vol. 93, No. 3, Mar. 2000, pp. 200–206.

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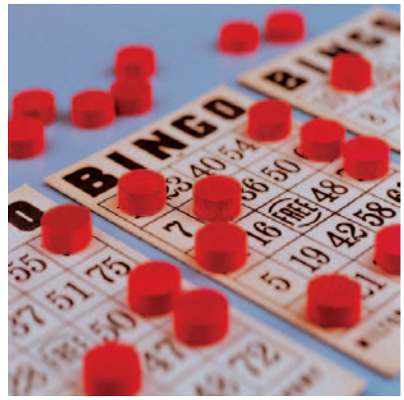
b. An L occurs when the first column and the bottom row are both covered. Find the probability that an L will occur in the fewest number of calls.

c. An X-out occurs when both diagonals are covered. Find the probability that an X-out occurs in the fewest number of calls.

d. If bingo cards are constructed so that column one has 5 of the numbers from 1 to 15, column two has 5 of the numbers from 16 to 30, column three has 4 of the numbers from 31 to 45, column four has 5 of the numbers from 46 to 60, and column five has 5 of the numbers from 61 to 75, how many different bingo cards could be constructed? (*Hint:* Order matters!)



8.4 BINOMIAL PROBABILITY

THINK ABOUT IT *What is the probability that 3 of 6 people prefer Diet Supercola over its competitors?*

This question involves an experiment that is repeated 6 times. Many probability problems are concerned with experiments in which an event is repeated many times. Other examples include finding the probability of getting 7 heads in 8 tosses of a coin, of hitting a target 6 times out of 6, and of finding 1 defective item in a sample of 15 items. Probability problems of this kind are called **Bernoulli trials** problems, or **Bernoulli processes,** named after the Swiss mathematician Jakob Bernoulli (1654–1705), who is well known for his work in probability theory. In each case, some outcome is designated a success, and any other outcome is considered a failure. This labeling is arbitrary, and does not necessarily have anything to do with real success or failure. Thus, if the probability of a success in a single trial is p, the probability of failure will be $1 - p$. A Bernoulli trials problem, or **binomial experiment,** must satisfy the following conditions.

BINOMIAL EXPERIMENT

1. The same experiment is repeated several times.
2. There are only two possible outcomes, success and failure.
3. The repeated trials are independent, so that the probability of success remains the same for each trial.

EXAMPLE 1 *Sleep*

The chance that an American falls asleep with the TV on at least three nights a week is $1/4$.* Suppose a researcher selects 5 Americans at random and is interested in the probability that all 5 are "TV sleepers."

Harper's Magazine, Mar. 1996, p. 13.

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FOR REVIEW

Recall that if A and B are independent events,

$$P(A \text{ and } B) = P(A)P(B).$$

Solution Here the experiment, selecting a person, is repeated 5 times. If selecting a TV sleeper is labeled a success, then getting a "non-TV sleeper" is labeled a failure. The 5 trials are almost independent. There is a very slight dependence; if, for example, the first person selected is a TV sleeper, then there is one less TV sleeper to choose from when we select the next person (assuming we never select the same person twice). When selecting a small sample out of a large population, however, the probability changes negligibly, so researchers consider such trials to be independent. Thus, the probability that all 5 in our sample are sleepers is

$$\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \left(\frac{1}{4}\right)^5 \approx .000977.$$

Now suppose the problem in Example 1 is changed to that of finding the probability that exactly 4 of the 5 people in the sample are TV sleepers. This outcome can occur in more than one way, as shown below, where s represents a success (a TV sleeper) and f represents a failure (a non-TV sleeper).

outcome 1:	s	s	s	s	f
outcome 2:	s	s	s	f	s
outcome 3:	s	s	f	s	s
outcome 4:	s	f	s	s	s
outcome 5:	f	s	s	s	s

Keep in mind that since the probability of success is $1/4$, the probability of failure is $1 - 1/4 = 3/4$. The probability, then, of each of these 5 outcomes is

$$\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right).$$

Since the 5 outcomes represent mutually exclusive events, add the 5 identical probabilities, which is equivalent to multiplying the above probability by 5. The result is

$$P(4 \text{ of the 5 people are TV sleepers}) = 5\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) = \frac{15}{4^5} \approx .01465.$$

In the same way, we can compute the probability of selecting 3 TV sleepers in our sample of 5. The probability of any one way of achieving 3 successes and 2 failures will be

$$\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2.$$

Rather than list all the ways of achieving 3 successes out of 5 trials, we will count this number using combinations. The number of ways to select 3 elements out of a set of 5 is $\binom{5}{3} = 5!/(2!\,3!) = 10$, giving

$$P(3 \text{ of the 5 people are TV sleepers}) = 10\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{90}{4^5} \approx .08789.$$

A similar argument works in the general case.

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BINOMIAL PROBABILITY

If p is the probability of success in a single trial of a binomial experiment, the probability of x successes and $n - x$ failures in n independent repeated trials of the experiment is

$$\binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}.$$

EXAMPLE 2 *Advertising*

The advertising agency that handles the Diet Supercola account believes that 40% of all consumers prefer this product over its competitors. Suppose a sample of 6 people is chosen. Assume that all responses are independent of each other. Find the probability of the following.

(a) Exactly 3 of the 6 people prefer Diet Supercola.

Solution Think of the 6 responses as 6 independent trials. A success occurs if a person prefers Diet Supercola. Then this is a binomial experiment with $p = P(\text{success}) = P(\text{prefer Diet Supercola}) = .4$. The sample is made up of 6 people, so $n = 6$. To find the probability that exactly 3 people prefer this drink, let $x = 3$ and use the formula in the box.

$$\begin{aligned} P(\text{exactly } 3) &= \binom{6}{3}(.4)^3(1-.4)^{6-3} \\ &= 20(.4)^3(.6)^3 \\ &= 20(.064)(.216) \\ &= .27648 \end{aligned}$$

(b) None of the 6 people prefer Diet Supercola.

Solution Let $x = 0$.

$$P(\text{exactly } 0) = \binom{6}{0}(.4)^0(1-.4)^6 = 1(1)(.6)^6 \approx .0467$$

EXAMPLE 3 *Coin Toss*

Find the probability of getting exactly 7 heads in 8 tosses of a fair coin.

Solution The probability of success (getting a head in a single toss) is $1/2$. The probability of a failure (getting a tail) is $1 - 1/2 = 1/2$. Thus,

$$P(7 \text{ heads in } 8 \text{ tosses}) = \binom{8}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^1 = 8\left(\frac{1}{2}\right)^8 = .03125.$$

EXAMPLE 4 *Defective Items*

Assuming that selection of items for a sample can be treated as independent trials, and that the probability that any 1 item is defective is .01, find the following.

(a) The probability of 1 defective item in a random sample of 15 items from a production line

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Solution Here, a "success" is a defective item. Since selecting each item for the sample is assumed to be an independent trial, the binomial probability formula applies. The probability of success (a defective item) is .01, while the probability of failure (an acceptable item) is .99. This makes

$$\begin{aligned} P(1 \text{ defective in 15 items}) &= \binom{15}{1}(.01)^1(.99)^{14} \\ &= 15(.01)(.99)^{14} \\ &\approx .130. \end{aligned}$$

(b) The probability of at most 1 defective item in a random sample of 15 items from a production line

Solution "At most 1" means 0 defective items or 1 defective item. Since 0 defective items is equivalent to 15 acceptable items,

$$P(0 \text{ defective}) = (.99)^{15} \approx .860.$$

Use the union rule, noting that 0 defective and 1 defective are mutually exclusive events, to get

$$\begin{aligned} P(\text{at most 1 defective}) &= P(0 \text{ defective}) + P(1 \text{ defective}) \\ &\approx .860 + .130 \\ &= .990. \end{aligned}$$

EXAMPLE 5 *Supermarket Scanners*

A survey by *Money* magazine found that supermarket scanners are overcharging customers at 30% of stores.*

(a) If you shop at 3 supermarkets that use scanners, what is the probability that you will be overcharged in at least one store?

Solution We can treat this as a binomial experiment, letting $n = 3$ and $p = .3$. At least 1 of 3 means 1 or 2 or 3. It will be simpler here to find the probability of being overcharged in none of the 3 stores, that is, $P(0 \text{ overcharges})$, and then find $1 - P(0 \text{ overcharges})$.

$$\begin{aligned} P(0 \text{ overcharges}) &= \binom{3}{0}(.3)^0(.7)^3 \\ &= 1(1)(.343) = .343 \\ P(\text{at least one}) &= 1 - P(0 \text{ overcharges}) \\ &= 1 - .343 = .657 \end{aligned}$$

(b) If you shop at 3 supermarkets that use scanners, what is the probability that you will be overcharged in at most one store?

Solution "At most one" means 0 or 1, so

$$\begin{aligned} P(0 \text{ or } 1) &= P(0) + P(1) \\ &= \binom{3}{0}(.3)^0(.7)^3 + \binom{3}{1}(.3)^1(.7)^2 \\ &= 1(1)(.343) + 3(.3)(.49) = .784. \end{aligned}$$

*O'Connell, Vanessa, "Don't Get Cheated by Supermarket Scanners," *Money,* Apr. 1993, pp. 132–138.

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The triangular array of numbers shown below is called **Pascal's triangle** in honor of the French mathematician Blaise Pascal (1623–1662), who was one of the first to use it extensively. The triangle was known long before Pascal's time and appears in Chinese and Islamic manuscripts from the eleventh century.

PASCAL'S TRIANGLE

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

The array provides a quick way to find binomial probabilities. The nth row of the triangle, where $n = 0, 1, 2, 3, \ldots,$ gives the coefficients $\binom{n}{r}$ for $r = 0,$ $1, 2, 3, \ldots, n$. For example, for $n = 4$, $1 = \binom{4}{0}$, $4 = \binom{4}{1}$, $6 = \binom{4}{2}$, and so on. Each number in the triangle is the sum of the two numbers directly above it. For example, in the row for $n = 4$, 1 is the sum of 1, the only number above it, 4 is the sum of 1 and 3, 6 is the sum of 3 and 3, and so on. Adding in this way gives the sixth row:

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1.$$

Notice that Pascal's triangle tells us, for example, that $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$ (that is, $4 + 6 = 10$). Using the combinations formula, it can be shown that, in general, $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$. This is left as an exercise.

EXAMPLE 6 *Pascal's Triangle*

Use Pascal's triangle to find the probability in Example 5 that if you shop at 6 supermarkets, at least 3 will overcharge you.

Solution The probability of success is .3. Since at least 3 means 3, 4, 5, or 6,

$$\begin{aligned}
P(\text{at least } 3) &= P(3) + P(4) + P(5) + P(6) \\
&= \binom{6}{3}(.3)^3(.7)^3 + \binom{6}{4}(.3)^4(.7)^2 \\
&\quad + \binom{6}{5}(.3)^5(.7)^1 + \binom{6}{6}(.3)^6(.7)^0.
\end{aligned}$$

Use the sixth row of Pascal's triangle for the combinations to get

$$\begin{aligned}
P(\text{at least } 3) &= 20(.3)^3(.7)^3 + 15(.3)^4(.7)^2 + 6(.3)^5(.7)^1 + 1(.3)^6(.7)^0 \\
&= .1852 + .0595 + .0102 + .0007 \\
&= .2556.
\end{aligned}$$

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EXAMPLE 7 *Independent Jury*

If each member of a 9-person jury acts independently of each other and makes the correct determination of guilt or innocence with probability .65, find the probability that the majority of jurors will reach a correct verdict.*

Solution

Method 1: Calculation by Hand

Since the jurors in this particular situation act independently, we can treat this as a binomial experiment. Thus, the probability that the majority of the jurors will reach the correct verdict is given by

$$\begin{aligned} P(\text{at least } 5) &= \binom{9}{5}(.65)^5(.35)^4 + \binom{9}{6}(.65)^6(.35)^3 + \binom{9}{7}(.65)^7(.35)^2 \\ &\quad + \binom{9}{8}(.65)^8(.35)^1 + \binom{9}{9}(.65)^9 \\ &= .2194 + .2716 + .2162 + .1004 + .0207 \\ &= .8283. \end{aligned}$$

Method 2: Graphing Calculator

Some graphing calculators provide binomial probabilities. On a TI-83/84 Plus, for example, the command `binompdf(9, .65, 5)`, found in the `DISTR` menu, gives .21939, which is the probability that $x = 5$. Alternatively, the command `binomcdf(9, .65, 4)` gives .17172 as the probability that 4 or fewer jurors will make the correct decision. Subtract .17172 from 1 to get .82828 as the probability that the majority of the jurors will make the correct decision. This value rounds to .8283, which is in agreement with Method 1. Often, Method 2 is more accurate than Method 1 due to the accumulation of rounding errors when doing successive calculations by hand.

Method 3: Spreadsheet

Some spreadsheets also provide binomial probabilities. In Microsoft Excel, for example, the command "`=BINOMDIST(5, 9, .65, 0)`" gives .21939, which is the probability that $x = 5$. Alternatively, the command "`=BINOMDIST(4, 9, .65, 1)`" gives .17172 as the probability that 4 or fewer jurors will make the correct decision. Subtract .17172 from 1 to get .82828 as the probability that the majority of the jurors will make the correct decision. This value agrees with the value found in Methods 1 and 2.

8.4 EXERCISES

Suppose that a family has 5 children. Also, suppose that the probability of having a girl is $1/2$. *Find the probabilities that the family has the following children.*

1. Exactly 2 girls and 3 boys
2. Exactly 3 girls and 2 boys
3. No girls
4. No boys
5. At least 4 girls
6. At least 3 boys
7. No more than 3 boys
8. No more than 4 girls

*Grofman, Bernard, "A Preliminary Model of Jury Decision Making as a Function of Jury Size, Effective Jury Decision Rule, and Mean Juror Judgmental Competence," *Frontiers in Economics,* 1979, pp. 98–110.

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A die is rolled 12 times. Find the probabilities of rolling the following.

9. Exactly 12 ones
10. Exactly 6 ones
11. Exactly 1 one
12. Exactly 2 ones
13. No more than 3 ones
14. No more than 1 one

A coin is tossed 6 times. Find the probabilities of getting the following.

15. All heads
16. Exactly 3 heads
17. No more than 3 heads
18. At least 3 heads

19. How do you identify a probability problem that involves a binomial experiment?

20. How is Pascal's triangle used to find probabilities?

21. Using the definition of combination in Section 8.2, prove that

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}.$$

(This is the formula underlying Pascal's triangle.)

In Exercises 22 and 23, argue that the use of binomial probabilities is not applicable and thus the probabilities that are computed are not correct.

22. In England, a woman was found guilty of smothering her two infant children. Much of the Crown's case against the lady was based on the testimony from a pediatrician who indicated that the chances of two crib deaths occurring in both siblings was only about 1 in 73 million. This number was calculated by assuming that the probability of a single crib death is 1 in 8500 and the probability of two crib deaths is 1 in 8500^2 (i.e., binomial).*

23. A contemporary radio station in Boston has a contest in which a caller is asked his or her date of birth. If the caller's date of birth, including the day, month, and year of birth, matches a predetermined date, the caller wins $1 million. Assuming that there were 36,525 days in the twentieth century and the contest was run 51 times on consecutive days, the probability that the grand prize will be won is

$$1 - \left(1 - \frac{1}{36{,}525}\right)^{51} \approx .0014.^{\dagger}$$

Applications

BUSINESS AND ECONOMICS

Management *The survey discussed in Example 5 also found that customers overpay for* 1 *out of every* 10 *items, on average. Suppose a customer purchases* 15 *items. Find the following probabilities.*

24. A customer overpays on 3 items.

25. A customer does not overpay for any item.

26. A customer overpays on at least one item.

27. A customer overpays on at least 2 items.

28. A customer overpays on at most 2 items.

Insurance Rating *The insurance industry has found that the probability is* .9 *that a life insurance applicant will qualify at the regular rates. Find the probabilities that of the next* 10 *applicants for life insurance, the following numbers will qualify at the regular rates.*

29. Exactly 10
30. Exactly 9
31. At least 9
32. Less than 8

*Watkins, Stephen J., "Conviction by Mathematical Error?," *British Medical Journal,* Vol. 320, No. 7226, Jan. 1, 2000, pp. 2–3.

†Snell, J. Laurie, "40-Million-Dollar Thursday," Chance News 9.04, Mar. 7–April 5, 2000.

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Personnel Screening *A company gives prospective workers a 6-question, multiple-choice test. Each question has 5 possible answers, so that there is a 1/5 or 20% chance of answering a question correctly just by guessing. Find the probabilities of getting the following results by chance.*

33. Exactly 2 correct answers

34. No correct answers

35. At least 4 correct answers

36. No more than 3 correct answers

37. *Customer Satisfaction* Over the last decade, 10% of all clients of J. K. Loss & Company have lost their life savings. Suppose a sample of 3 of the current clients of the firm is chosen. Assuming independence, find the probability that exactly 1 of the 3 clients will lose everything.

Quality Control *A factory tests a random sample of 20 transistors for defective transistors. The probability that a particular transistor will be defective has been established by past experience as .05.*

38. What is the probability that there are no defective transistors in the sample?

39. What is the probability that the number of defective transistors in the sample is at most 2?

40. *Quality Control* The probability that a certain machine turns out a defective item is .05. Find the probabilities that in a run of 75 items, the following results are obtained.

a. Exactly 5 defective items

b. No defective items

c. At least 1 defective item

41. *Survey Results* A company is taking a survey to find out whether people like its product. Its last survey indicated that 70% of the population like the product. Based on that, in a sample of 58 people, find the probabilities of the following.

a. All 58 like the product.

b. From 28 to 30 (inclusive) like the product.

42. *Pecans* Pecan producers blow air through the pecans so that the lighter ones are blown out. The lighter-weight pecans are generally bad and the heavier ones tend to be better. These "blow outs" and "good nuts" are often sold to tourists along the highway. Suppose 60% of the "blow outs" are good, and 80% of the "good nuts" are good.*

a. What is the probability that if you crack and check 20 "good nuts" you will find 8 bad ones?

b. What is the probability that if you crack and check 20 "blow outs" you will find 8 bad ones?

c. If we assume that 70% of the roadside stands sell "good nuts," and that out of 20 nuts we find 8 that are bad, what is the probability that the nuts are "blow outs"?

LIFE SCIENCES

Drug Effectiveness *A new drug cures 70% of the people taking it. Suppose 20 people take the drug; find the probabilities of the following.*

43. Exactly 18 people are cured.

44. Exactly 17 people are cured.

45. At least 17 people are cured.

46. At least 18 people are cured.

Births of Twins *The probability that a birth will result in twins is .012. Assuming independence (perhaps not a valid assumption), what are the probabilities that out of 100 births in a hospital, there will be the following numbers of sets of twins?*

47. Exactly 2 sets of twins

48. At most 2 sets of twins

Vitamin A Deficiency *Six mice from the same litter, all suffering from a vitamin A deficiency, are fed a certain dose of carrots. If the probability of recovery under such treatment is .70, find the probabilities of the following results.*

49. None of the mice recover.

50. Exactly 3 of the 6 mice recover.

51. All of the mice recover.

52. No more than 3 mice recover.

53. *Effects of Radiation* In an experiment on the effects of a radiation dose on cells, a beam of radioactive particles is aimed at a group of 10 cells. Find the probability that 8 of the cells will be hit by the beam, if the probability that any single cell will be hit is .6. (Assume independence.)

*Submitted by Professor Irvin R. Hentzel, Iowa State University.

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54. *Effects of Radiation* The probability of a mutation of a given gene under a dose of 1 roentgen of radiation is approximately 2.5×10^{-7}. What is the probability that in 10,000 genes, at least 1 mutation occurs?

55. *Drug Side Effects* A new drug being tested causes a serious side effect in 5 out of 100 patients. What is the probability that no side effects occur in a sample of 10 patients taking the drug?

56. *Flu Inoculations* A flu vaccine has a probability of 80% of preventing a person who is inoculated from getting the flu. A county health office inoculates 83 people. Find the probabilities of the following.

a. Exactly 10 of the people inoculated get the flu.

b. No more than 4 of the people inoculated get the flu.

c. None of the people inoculated get the flu.

57. *Color Blindness* The probability that a male will be color-blind is .042. Find the probabilities that in a group of 53 men, the following will be true.

a. Exactly 5 are color-blind.

b. No more than 5 are color-blind.

c. At least 1 is color-blind.

58. *Pharmacology* In placebo-controlled trials of Pravachol®, a drug that is prescribed to lower cholesterol, 7.3% of the patients who were taking the drug experienced nausea/vomiting, whereas 7.1% of the patients who were taking the placebo experienced nausea/vomiting.*

a. If 100 patients who are taking Pravachol® are selected, what is the probability that 10 or more will experience nausea/vomiting?

b. If a second group of 100 patients receives a placebo, what is the probability that 10 or more will experience nausea/vomiting?

c. Since 7.3% is larger than 7.1%, do you believe that the Pravachol® causes more people to experience nausea/vomiting than a placebo? Explain.

59. *Genetic Fingerprinting* The use of DNA has become an integral part of many court cases. When DNA is extracted from cells and body fluids, genetic information is represented by bands of information, which look similar to a bar code at a grocery store. It is generally accepted that in unrelated people, the probability of a particular band matching is 1 in 4.†

a. If 5 bands are compared in unrelated people, what is the probability that all 5 of the bands match? (Express your answer in terms of "1 chance in ?".)

b. If 20 bands are compared in unrelated people, what is the probability that all 20 of the bands match? (Express your answer in terms of "1 chance in ?".)

c. If 20 bands are compared in unrelated people, what is the probability that 16 or more bands match? (Express your answer in terms of "1 chance in ?".)

d. If you were deciding paternity and there were 16 matches out of 20 bands compared, would you believe that the person being tested was the father? Explain.

SOCIAL SCIENCES

60. *Women Working* A recent study found that 33% of women would prefer to work part-time rather than full-time if money were not a concern.‡ Find the probability that if 10 women are selected at random, at least 3 of them would prefer to work part-time.

Testing In a 10-question, multiple-choice biology test with 5 choices for each question, an unprepared student guesses the answer to each item. Find the probabilities of the following results.

61. Exactly 6 correct answers

62. Exactly 7 correct answers

63. At least 8 correct answers

64. Fewer than 8 correct answers

65. *Community College Population* According to the state of California, 33% of all state community college students belong to ethnic minorities. Find the probabilities of the following results in a random sample of 10 California community college students.

a. Exactly 2 belong to an ethnic minority.

b. Three or fewer belong to an ethnic minority.

c. Exactly 5 do not belong to an ethnic minority.

d. Six or more do not belong to an ethnic minority.

66. *Cheating* According to a poll conducted by *U.S. News and World Report*, 84% of college students believe they need to cheat to get ahead in the world today.§

*Advertisement in *Time,* July 17, 2000, for Pravachol®, developed and marketed by Bristol-Myers Squibb Company.

†"Genetic Fingerprinting Worksheet," Centre for Innovation in Mathematics Teaching, http://www.ex.ac.uk/cimt/resource/fgrprnts.htm.

‡Ferraro, Cathleen, "Feelings of the Working Women," *The Sacramento Bee,* May 11, 1995, pp. A1, A22.

§Kleiner, Carolyn and Mary Lord, "The Cheating Game," *U.S. News and World Report,* Nov. 22, 1999, pp. 55–66.

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a. Do the results of this poll indicate that 84% of all college students cheat? Explain.

b. If this result is accurate and 100 college students are asked if they believe that cheating is necessary to get ahead in the world, what is the probability that 90 or more of the students will answer affirmatively to the question?

67. ***Education*** In the "Numbers" section of a recent *Time* magazine, it was reported that 15.2% of low-birth-weight babies graduate from high school by age 19. On the other hand, it was reported that 57.5% of the normal-birth-weight siblings graduated from high school.*

a. If 40 low-birth-weight babies were tracked through high school, what is the probability that fewer than 15 will graduate from high school by age 19?

b. What are some of the factors that may contribute to the wide difference in high school success between these siblings? Do you believe that low birth weight is the primary cause of the difference? What other information do you need to better answer these questions?

8.5 PROBABILITY DISTRIBUTIONS; EXPECTED VALUE

THINK ABOUT IT *What is the expected value of winning a prize for someone who buys one ticket in a raffle?*

In this section we shall see that the *expected value* of a probability distribution is a type of average. Probability distributions were introduced briefly in the chapter on Sets and Probability. Now we take a more complete look at probability distributions. A probability distribution depends on the idea of a *random variable,* so we begin with that.

Random Variables Suppose that the shipping manager at a company receives a package of one dozen computer monitors, of which, unknown to him, three are broken. He checks four of the monitors at random to see how many are broken in his sample of 4. The answer, which we will label x, is one of the numbers 0, 1, 2, or 3. Since the value of x is random, x is called a random variable.

RANDOM VARIABLE

A **random variable** is a function that assigns a real number to each outcome of an experiment.

Probability Distribution In the example with the shipping manager, we can calculate the probability that 0, 1, 2, or 3 monitors in his sample of 4 are broken using the methods of Section 8.3. There are 3 broken monitors and 9 unbroken monitors, so the number of ways of choosing 0 broken monitors (which implies 4 unbroken monitors) is $\binom{3}{0}\binom{9}{4}$. The number of ways of choosing a

*"Numbers," *Time,* July 17, 2000, p. 21.

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sample of 4 monitors is $\binom{12}{4}$. Therefore, the probability of choosing 0 broken monitors is

$$P(0) = \frac{\binom{3}{0}\binom{9}{4}}{\binom{12}{4}} = \frac{1\left(\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}\right)}{\left(\frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1}\right)} = \frac{126}{495} = \frac{14}{55}.$$

Similarly, the probability of choosing 1 broken monitor is

$$P(1) = \frac{\binom{3}{1}\binom{9}{3}}{\binom{12}{4}} = \frac{3 \cdot 84}{495} = \frac{252}{495} = \frac{28}{55}.$$

The probability of choosing 2 broken monitors is

$$P(2) = \frac{\binom{3}{2}\binom{9}{2}}{\binom{12}{4}} = \frac{3 \cdot 36}{495} = \frac{108}{495} = \frac{12}{55}.$$

The probability of choosing 3 broken monitors is

$$P(3) = \frac{\binom{3}{3}\binom{9}{1}}{\binom{12}{4}} = \frac{1 \cdot 9}{495} = \frac{9}{495} = \frac{1}{55}.$$

The results can be put in a table.

x	0	1	2	3
$P(x)$	14/55	28/55	12/55	1/55

Such a table that lists the possible values of a random variable, together with the corresponding probabilities, is called a **probability distribution.** The sum of the probabilities in a probability distribution must always equal 1. (The sum in some distributions may vary slightly from 1 because of rounding.)

Instead of writing the probability distribution as a table, we could write the same information as a set of ordered pairs:

$$\{(0, 14/55),(1, 28/55),(2, 12/55),(3, 1/55)\}.$$

There is just one probability for each value of the random variable. Thus, a probability distribution defines a function, called a **probability distribution function,** or simply a **probability function.** We shall use the terms "probability distribution" and "probability function" interchangeably.

The information in a probability distribution is often displayed graphically as a special kind of bar graph called a **histogram.** The bars of a histogram all have the same width, usually 1. The heights of the bars are determined by the probabil-

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ities. A histogram for the data in the table above is given in Figure 8. A histogram shows important characteristics of a distribution that may not be readily apparent in tabular form, such as the relative sizes of the probabilities and any symmetry in the distribution.

The area of the bar above $x = 0$ in Figure 8 is the product of 1 and 14/55, or $1 \cdot 14/55 = 14/55$. Since each bar has a width of 1, its area is equal to the probability that corresponds to that value of x. The probability that a particular value will occur is thus given by the area of the appropriate bar of the graph. For example, the probability that one or more monitors is broken is the sum of the areas for $x = 1$, $x = 2$, and $x = 3$. This area, shown in pink in Figure 9, corresponds to 41/55 of the total area, since

$$\begin{aligned} P(x \geq 1) &= P(x = 1) + P(x = 2) + P(x = 3) \\ &= 28/55 + 12/55 + 1/55 + 41/55. \end{aligned}$$

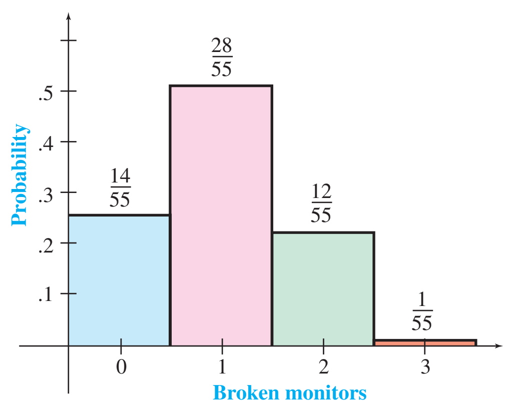

FIGURE 8

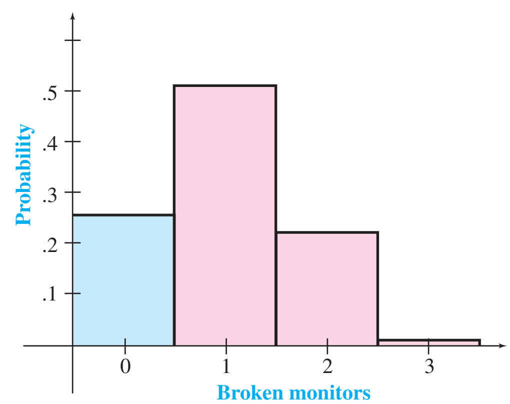

FIGURE 9

EXAMPLE 1 *Probability Distributions*

(a) Give the probability distribution for the number of heads showing when two coins are tossed.

Solution Let x represent the random variable "number of heads." Then x can take on the values 0, 1, or 2. Now find the probability of each outcome. The results are shown in the table with Figure 10.

x	0	1	2
$P(x)$	1/4	1/2	1/4

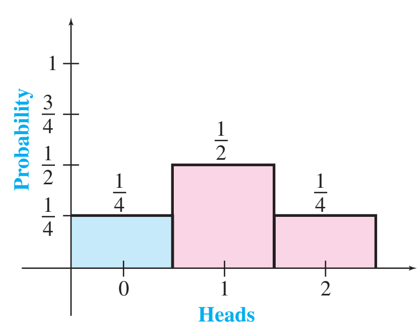

FIGURE 10

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(b) Draw a histogram for the distribution in the table. Find the probability that at least one coin comes up heads.

Solution The histogram is shown in Figure 10. The portion in red represents

$$P(x \geq 1) = P(x = 1) + P(x = 2) = \frac{3}{4}.$$

Expected Value

In working with probability distributions, it is useful to have a concept of the typical or average value that the random variable takes on. In Example 1, for instance, it seems reasonable that, on the average, one head shows when two coins are tossed. This does not tell what will happen the next time we toss two coins; we may get two heads, or we may get none. If we tossed two coins many times, however, we would expect that, in the long run, we would average about one head for each toss of two coins.

A way to solve such problems in general is to imagine flipping two coins 4 times. Based on the probability distribution in Example 1, we would expect that 1 of the 4 times we would get 0 heads, 2 of the 4 times we would get 1 head, and 1 of the 4 times we would get 2 heads. The total number of heads we would get, then, is

$$0 \cdot 1 + 1 \cdot 2 + 2 \cdot 1 = 4.$$

The expected numbers of heads per toss is found by dividing the total number of heads by the total number of tosses, or

$$\frac{0 \cdot 1 + 1 \cdot 2 + 2 \cdot 1}{4} = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1.$$

Notice that the expected number of heads turns out to be the sum of the three values of the random variable x multiplied by their corresponding probabilities. We can use this idea to define the *expected value* of a random variable as follows.

EXPECTED VALUE

Suppose the random variable x can take on the n values $x_1, x_2, x_3, \ldots, x_n$. Also, suppose the probabilities that these values occur are, respectively, $p_1, p_2, p_3, \ldots, p_n$. Then the **expected value** of the random variable is

$$E(x) = x_1p_1 + x_2p_2 + x_3p_3 + \cdots + x_np_n.$$

EXAMPLE 2 *Computer Monitors*

In the example with the computer monitors, find the expected number of broken monitors that the shipping manager finds.

Solution Using the values in the first table in this section and the definition of expected value, we find that

$$E(x) = 0 \cdot \frac{14}{55} + 1 \cdot \frac{28}{55} + 2 \cdot \frac{12}{55} + 3 \cdot \frac{1}{55} = 1.$$

On the average, the shipping manager will find 1 broken monitor in the sample of 4. On reflection, this seems natural; 3 of the 12 monitors, or 1/4 of the total, are broken. We should expect, then, that 1/4 of the sample of 4 monitors are broken.

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Physically, the expected value of a probability distribution represents a balance point. If we think of the histogram in Figure 8 as a series of weights with magnitudes represented by the heights of the bars, then the system would balance if supported at the point corresponding to the expected value.

EXAMPLE 3 *Symphony Orchestra*

Suppose a local symphony decides to raise money by raffling a microwave oven worth \$400, a dinner for two worth \$80, and 2 books worth \$20 each. A total of 2000 tickets are sold at \$1 each. Find the expected value of winning for a person who buys one ticket in the raffle.

Solution Here the random variable represents the possible amounts of net winnings, where net winnings = amount won − cost of ticket. The net winnings of the person winning the oven are \$400 (amount won) − \$1 (cost of ticket) = \$399. The net winnings for each losing ticket are \$0 − \$1 = −\$1.

The net winnings of the various prizes, as well as their respective probabilities, are shown in the table below. The probability of winning \$19 is 2/2000 because there are 2 prizes worth \$20. We have not reduced the fractions in order to keep all the denominators equal. Because there are 4 winning tickets, there are 1996 losing tickets, so the probability of winning −\$1 is 1996/2000.

x	\$399	\$79	\$19	−\$1
$P(x)$	1/2000	1/2000	2/2000	1996/2000

The expected winnings for a person buying one ticket are

$$\begin{aligned} 399\left(\frac{1}{2000}\right) + 79\left(\frac{1}{2000}\right) + 19\left(\frac{2}{2000}\right) + (-1)\left(\frac{1996}{2000}\right) &= -\frac{1480}{2000} \\ &= -.74. \end{aligned}$$

On the average, a person buying one ticket in the raffle will lose \$.74, or 74¢.

It is not possible to lose 74¢ in this raffle: either you lose \$1, or you win a prize worth \$400, \$80, or \$20, minus the \$1 you pay to play. But if you bought tickets in many such raffles over a long period of time, you would lose 74¢ per ticket on the average. It is important to note that the expected value of a random variable may be a number that can never occur in any one trial of the experiment.

NOTE An alternative way to compute expected value in this and other examples is to calculate the expected amount won and then subtract the cost of the ticket afterward. The amount won is either \$400 (with probability 1/2000), \$80 (with probability 1/2000), \$20 (with probability 2/2000), or \$0 (with probability 1996/2000). The expected winnings for a person buying one ticket are then

$$\begin{aligned} 400\left(\frac{1}{2000}\right) + 80\left(\frac{1}{2000}\right) + 20\left(\frac{2}{2000}\right) + 0\left(\frac{1996}{2000}\right) - 1 &= -\frac{1480}{2000} \\ &= -.74. \end{aligned}$$ ■

EXAMPLE 4 *Friendly Wager*

Each day Donna and Mary toss a coin to see who buys coffee (80 cents a cup). One tosses and the other calls the outcome. If the person who calls the outcome is correct, the other buys the coffee; otherwise the caller pays. Find Donna's expected winnings.

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Solution Assume that an honest coin is used, that Mary tosses the coin, and that Donna calls the outcome. The possible results and corresponding probabilities are shown below.

	Possible Results			
Result of Toss	Heads	Heads	Tails	Tails
Call	Heads	Tails	Heads	Tails
Caller Wins?	Yes	No	No	Yes
Probability	1/4	1/4	1/4	1/4

Donna wins an 80¢ cup of coffee whenever the results and calls match, and she loses an 80¢ cup when there is no match. Her expected winnings are

$$(.80)\left(\frac{1}{4}\right) + (-.80)\left(\frac{1}{4}\right) + (-.80)\left(\frac{1}{4}\right) + (.80)\left(\frac{1}{4}\right) = 0.$$

On the average, over the long run, Donna neither wins nor loses.

A game with an expected value of 0 (such as the one in Example 4) is called a **fair game.** Casinos do not offer fair games. If they did, they would win (on the average) \$0, and have a hard time paying the help! Casino games have expected winnings for the house that vary from 1.5 cents per dollar to 60 cents per dollar. Exercises 43–48 at the end of the section ask you to find the expected winnings for certain games of chance.

The idea of expected value can be very useful in decision making, as shown by the next example.

EXAMPLE 5 *Life Insurance*

At age 50, you receive a letter from Mutual of Mauritania Insurance Company. According to the letter, you must tell the company immediately which of the following two options you will choose: take \$20,000 at age 60 (if you are alive, \$0 otherwise) or \$30,000 at age 70 (again, if you are alive, \$0 otherwise). Based *only* on the idea of expected value, which should you choose?

Solution Life insurance companies have constructed elaborate tables showing the probability of a person living a given number of years into the future. From a recent such table, the probability of living from age 50 to 60 is .88, while the probability of living from age 50 to 70 is .64. The expected values of the two options are given below.

$$\text{First option: } (20{,}000)(.88) + (0)(.12) = 17{,}600$$
$$\text{Second option: } (30{,}000)(.64) + (0)(.36) = 19{,}200$$

Based strictly on expected values, choose the second option.

EXAMPLE 6 *Bachelor's Degrees*

According to the National Center for Education Statistics, 76.7% of those earning bachelor's degrees in education in the United States in 2000–2001 were

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female.* Suppose 5 holders of bachelor's degrees in education from 2000–2001 are picked at random.

(a) Find the probability distribution for the number that are female.

Solution We first note that each of the 5 people in the sample is either female (with probability .767) or male (with probability .233). As in the previous section, we may assume that the probability for each member of the sample is independent of that of any other. Such a situation is described by binomial probability with $n = 5$ and $p = .767$, for which we use the binomial probability formula

$$\binom{n}{x} \cdot p^x \cdot (1 - p)^{n-x},$$

where x is the number of females in the sample. For example,

$$P(x = 0) = \binom{5}{0}(.767)^0(.233)^5 \approx .0007.$$

Similarly, we could calculate the probability that x is any value from 0 to 5, resulting in the following probability distribution (with all probabilities rounded to four places).

x	0	1	2	3	4	5
$P(x)$	.0007	.0113	.0744	.2450	.4032	.2654

(b) Find the expected number of females in the sample of 5 people.

Solution Using the formula for expected value, we have

$$E(x) = 0(.0007) + 1(.0113) + 2(.0744) + 3(.2450) + 4(.4032) + 5(.2654) = 3.835.$$

On the average, 3.835 of the people in the sample of 5 will be female.

There is another way to get the answer in part (b) of the previous example. Because 76.7% of those earning bachelor's degrees in education in the United States in 2000–2001 are female, it is reasonable to expect 76.7% of our sample to be female. Thus, 76.7% of 5 is $5(.767) = 3.835$. Notice that what we have done is to multiply n by p. It can be shown that this method always gives the expected value for binomial probability.

EXPECTED VALUE FOR BINOMIAL PROBABILITY

For binomial probability, $E(x) = np$. In other words, the expected number of successes is the number of trials times the probability of success in each trial.

*http://nces.ed.gov//programs/digest/d02/tables/dt265.asp.

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EXAMPLE 7 *Female Children*

Suppose a family has 3 children.

(a) Find the probability distribution for the number of girls.

Solution Assuming girls and boys are equally likely, the probability distribution is binomial with $n = 3$ and $p = 1/2$. Letting x be the number of girls in the formula for binomial probability, we find, for example,

$$P(x = 0) = \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

The other values are found similarly, and the results are shown in the following table.

x	0	1	2	3
$P(x)$	1/8	3/8	3/8	1/8

We can verify this by noticing that in the sample space S of all 3-child families, there are eight equally likely outcomes: $S = \{ggg, ggb, gbg, gbb, bgg, bgb, bbg, bbb\}$. One of the outcomes has 0 girls, three have 1 girl, three have 2 girls, and one has 3 girls.

(b) Find the expected number of girls in a 3-child family using the distribution from part (a).

Solution Using the formula for expected value, we have

$$\text{Expected number of girls} = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$
$$= \frac{12}{8} = 1.5.$$

On average, a 3-child family will have 1.5 girls. This result agrees with our intuition that, on the average, half the children born will be girls.

(c) Find the expected number of girls in a 3-child family using the formula for expected value for binomial probability.

Solution Using the formula $E(x) = np$ with $n = 3$ and $p = 1/2$, we have

$$\text{Expected number of girls} = 3\left(\frac{1}{2}\right) = 1.5.$$

This agrees with our answer from part (b), as it must.

8.5 EXERCISES

For each experiment described below, let x determine a random variable, and use your knowledge of probability to prepare a probability distribution.

1. Four coins are tossed, and the number of heads is noted.
2. Two dice are rolled, and the total number of points is recorded.

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3. Three cards are drawn from a deck. The number of aces is counted.
4. Two balls are drawn from a bag in which there are 4 white balls and 2 black balls. The number of black balls is counted.

Draw a histogram for the following, and shade the region that gives the indicated probability.

5. Exercise 1; $P(x \leq 2)$
6. Exercise 2; $P(x \geq 11)$
7. Exercise 3; $P(\text{at least one ace})$
8. Exercise 4; $P(\text{at least one black ball})$

Find the expected value for each random variable.

9.

x	2	3	4	5
$P(x)$	.1	.4	.3	.2

10.

y	4	6	8	10
$P(y)$	.4	.4	.05	.15

11.

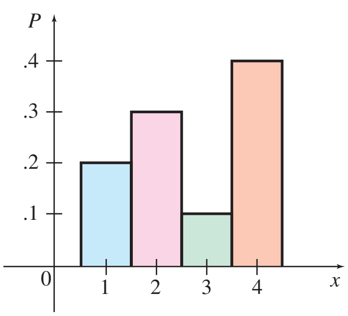
z	9	12	15	18	21
$P(z)$	.14	.22	.36	.18	.10

12.

x	30	32	36	38	44
$P(x)$	.31	.30	.29	.06	.04

Find the expected value for the random variable x having the probability function shown in each graph.

13.



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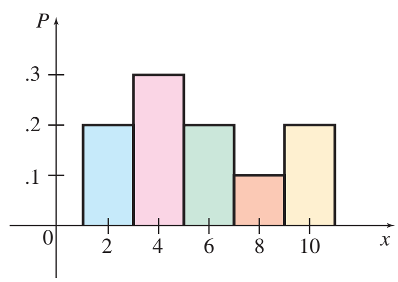

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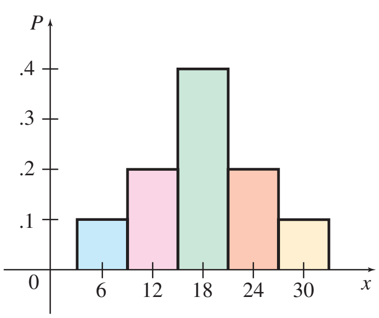

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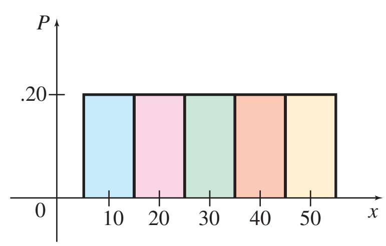

17. For the game in Example 4, find Mary's expected winnings. Is it a fair game?
18. Suppose one day Mary brings a 2-headed coin and uses it to toss for the coffee. Since Mary tosses, Donna calls.
    a. Is this still a fair game?
    b. What is Donna's expected gain if she calls heads?
    c. What is Donna's expected gain if she calls tails?

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Solve each exercise. Many of these exercises require the use of combinations.

19. Suppose 3 marbles are drawn from a bag containing 3 yellow and 4 white marbles.

a. Draw a histogram for the number of yellow marbles in the sample.

b. What is the expected number of yellow marbles in the sample?

20. Suppose 5 apples in a barrel of 25 apples are known to be rotten.

a. Draw a histogram for the number of rotten apples in a sample of 2 apples.

b. What is the expected number of rotten apples in a sample of 2 apples?

21. Suppose a die is rolled 4 times.

a. Find the probability distribution for the number of times 1 is rolled.

b. What is the expected number of times 1 is rolled?

22. A delegation of 3 is selected from a city council made up of 5 liberals and 4 conservatives.

a. What is the expected number of liberals in the delegation?

b. What is the expected number of conservatives in the delegation?

23. From a group of 2 women and 5 men, a delegation of 2 is selected. Find the expected number of women in the delegation.

24. In a club with 20 senior and 10 junior members, what is the expected number of junior members on a 3-member committee?

25. If 2 cards are drawn at one time from a deck of 52 cards, what is the expected number of diamonds?

26. Suppose someone offers to pay you \$5 if you draw 2 diamonds in the game in Exercise 25. He says that you should pay 50 cents for the chance to play. Is this a fair game?

27. Your friend missed class the day probability distributions were discussed. How would you explain probability distribution to him?

28. Explain what expected value means in your own words.

Applications

BUSINESS AND ECONOMICS

29. ***Complaints*** A local used-car dealer gets complaints about his cars as shown in the table below.

Find the expected number of complaints per day.

Number of Complaints per Day	0	1	2	3	4	5	6
Probability	.01	.05	.15	.26	.33	.14	.06

30. ***Payout on Insurance Policies*** An insurance company has written 100 policies of \$10,000, 500 of \$5000, and 1000 of \$1000 for people of age 20. If experience shows that the probability that a person will die at age 20 is .001, how much can the company expect to pay out during the year the policies were written?

31. ***Rating Sales Accounts*** Levi Strauss and Company* uses expected value to help its salespeople rate their accounts. For each account, a salesperson estimates potential additional volume and the probability of getting it. The product of these figures gives the expected value of the potential, which is added to the existing volume. The totals are then classified as A, B, or C, as follows: \$40,000 or below, class C; from \$40,000 up to and including \$55,000, class B; above \$55,000, class A. Complete the table on the next page for one salesperson.

*This example was supplied by James McDonald, Levi Strauss and Company, San Francisco.

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Account Number	Existing Volume	Potential Additional Volume	Probability of Getting It	Expected Value of Potential	Existing Volume + Expected Value of Potential	Class
1	$15,000	$10,000	.25	$2500	$17,500	C
2	$40,000	$0	—	—	$40,000	C
3	$20,000	$10,000	.20			
4	$50,000	$10,000	.10			
5	$5000	$50,000	.50			
6	$0	$100,000	.60			
7	$30,000	$20,000	.80			

32. ***Pecans*** Refer to Exercise 42 in Section 8.4. Suppose that 60% of the pecan "blow outs" are good, and 80% of the "good nuts" are good.

a. If you purchase 50 pecans, what is the expected number of good nuts you will find if you purchase "blow outs"?

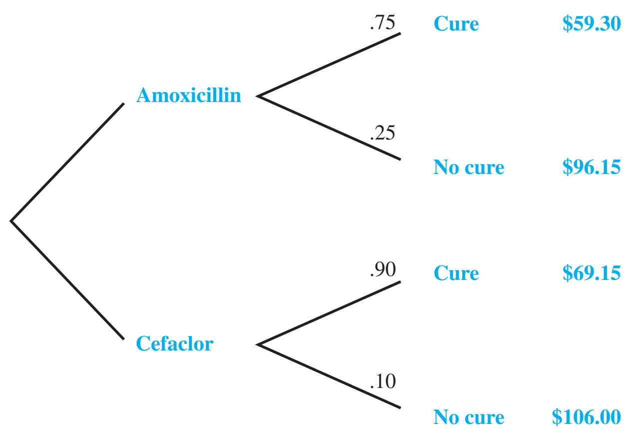
b. If you purchase 50 pecans, what is the expected number of bad nuts you will find if you have purchased "good nuts"?

LIFE SCIENCES

33. ***Animal Offspring*** In a certain animal species, the probability that a healthy adult female will have no offspring in a given year is .31, while the probabilities of 1, 2, 3, or 4 offspring are, respectively, .21, .19, .17, and .12. Find the expected number of offspring.

34. ***Ear Infections*** Otitis media, or middle ear infection, is initially treated with an antibiotic. Researchers have compared two antibiotics, amoxicillin and cefaclor, for their cost effectiveness. Amoxicillin is inexpensive, safe, and effective. Cefaclor is also safe. However, it is considerably more expensive and it is generally more effective. Use the tree diagram in the next column (where the costs are estimated as the total cost of medication, office visit, ear check, and hours of lost work) to answer the following.*

a. Find the expected cost of using each antibiotic to treat a middle ear infection.



b. To minimize the total expected cost, which antibiotic should be chosen?

PHYSICAL SCIENCES

35. ***Seeding Storms*** One of the few methods that can be used in an attempt to cut the severity of a hurricane is to *seed* the storm. In this process, silver iodide crystals are dropped into the storm. Unfortunately, silver iodide crystals sometimes cause the storm to *increase* its speed. Wind speeds may also increase or decrease even with no seeding. Use the tree diagram on the next page to answer the following.†

a. Find the expected amount of damage under each option, "seed" and "do not seed."

*Weiss, Jeffrey and Shoshana Melman, based on "Cost Effectiveness in the Choice of Antibiotics for the Initial Treatment of Otitis Media in Children: A Decision Analysis Approach," *Journal of Pediatric Infectious Disease,* Vol. 7, No. 1, 1988, pp. 23–26.

†Based on the probabilities and amounts of property damage in the tree diagram for Exercise 35 are from Howard, R. A., J. E. Matheson, and D. W. North, "The Decision to Seed Hurricanes," *Science,* Vol. 176, No. 16, June 1972, pp. 1191–1202. Copyright © 1972 by the American Association for the Advancement of Science.

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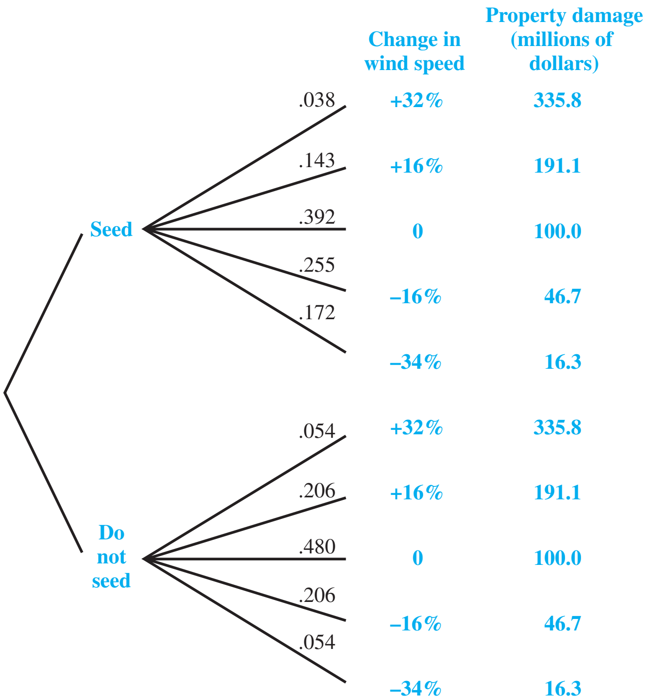

b. To minimize total expected damage, what option should be chosen?

SOCIAL SCIENCES

36. ***Cheating*** Recall from Exercise 66 in Section 8.4 that a poll conducted by *U.S. News and World Report* reported that 84% of college students believe they need to cheat to get ahead in the world today.* If 500 college students were surveyed, how many would you expect to say that they need to cheat to get ahead in the world today?

37. ***Education*** Recall from Exercise 67 in Section 8.4 that a *Time* magazine "Numbers" section reported that 15.2% of low-birth-weight babies graduate from high school by age 19.† If 250 low-birth-weight babies are followed through high school, how many would you expect to graduate from high school?

GENERAL INTEREST

38. ***Golf Tournament*** At the end of play in a major golf tournament, two players, an "old pro" and a "new kid," are tied. Suppose the first prize is \$80,000 and second prize is \$20,000. Find the expected winnings for the old pro if

a. both players are of equal ability;

b. the new kid will freeze up, giving the old pro a $3/4$ chance of winning.

39. ***Cats*** Kimberly Workman has four cats: Riley, Abby, Beastie, and Sylvester. Each cat has a 30% probability of climbing into the chair in which Kimberly is sitting, independent of how many cats are already in the chair with Kimberly.

a. Find the probability distribution for the number of cats in the chair with Kimberly.

b. Find the expected number of cats in the chair with Kimberly using the probability distribution in part a.

c. Find the expected number of cats in the chair with Kimberly using the formula for expected value of the binomial distribution.

40. ***Postal Service*** Mr. Statistics (a feature in *Fortune* magazine) investigated the claim of the U.S. Postal Service that 83% of first class mail in New York City arrives by the next day.‡ (The figure is 87% nationwide.) He mailed a letter to himself on 10 consecutive days; only 4 were delivered by the next day.

a. Find the probability distribution for the number of letters delivered by the next day if the overall probability of next-day delivery is 83%.

b. Using your answer to part a, find the probability that 4 or fewer out of 10 letters would be delivered by the next day.

c. Based on your answer to part b, do you think it is likely that the 83% figure is accurate? Explain.

d. Find the number of letters out of 10 that you would expect to be delivered by the next day if the 83% figure is accurate.

41. ***Raffle*** A raffle offers a first prize of \$100 and 2 second prizes of \$40 each. One ticket costs \$1, and 500 tickets are sold. Find the expected winnings for a person who buys 1 ticket. Is this a fair game?

42. ***Raffle*** A raffle offers a first prize of \$1000, 2 second prizes of \$300 each, and 20 third prizes of \$10 each. If 10,000 tickets are sold at 50¢ each, find the expected winnings for a person buying 1 ticket. Is this a fair game?

*Kleiner, Carolyn and Mary Lord, "The Cheating Game," *U.S. News and World Report,* Nov. 22, 1999, pp. 55–66.
†"Numbers," *Time,* July 17, 2000, p. 21.
‡Seligman, Daniel, "Ask Mr. Statistics," *Fortune,* July 24, 1995, pp. 170–171.

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Find the expected winnings for the games of chance described in Exercises 43–48.

43. ***Lottery*** A state lottery requires you to choose 4 cards from an ordinary deck: 1 heart, 1 club, 1 diamond, and 1 spade in that order from the 13 cards in each suit. If all four choices are selected by the lottery, you win \$5000. It costs \$1 to play.

44. ***Lottery*** If exactly 3 of the 4 choices in Exercise 43 are selected, the player wins \$200. (Ignore the possibility that all 4 choices are selected. It still costs \$1 to play.)

45. ***Roulette*** In one form of roulette, you bet \$1 on "even." If 1 of the 18 even numbers comes up, you get your dollar back, plus another one. If 1 of the 20 noneven (18 odd, 0, and 00) numbers comes up, you lose your dollar.

46. ***Roulette*** In another form of roulette, there are only 19 noneven numbers (no 00).

47. ***Numbers*** *Numbers* is a game in which you bet \$1 on any three-digit number from 000 to 999. If your number comes up, you get \$500.

48. ***Keno*** In one form of the game *Keno*, the house has a pot containing 80 balls, each marked with a different number from 1 to 80. You buy a ticket for \$1 and mark one of the 80 numbers on it. The house then selects 20 numbers at random. If your number is among the 20, you get \$3.20 (for a net winning of \$2.20).

49. ***Contests*** A magazine distributor offers a first prize of \$100,000, two second prizes of \$40,000 each, and two third prizes of \$10,000 each. A total of 2,000,000 entries are received in the contest. Find the expected winnings if you submit one entry to the contest. If it would cost you 50¢ in time, paper, and stamps to enter, would it be worth it?

50. ***Contests*** A contest at a fast-food restaurant offered the following cash prizes and probabilities of winning on one visit.

Prize	Probability
\$100,000	1/176,402,500
\$25,000	1/39,200,556
\$5000	1/17,640,250
\$1000	1/1,568,022
\$100	1/282,244
\$5	1/7056
\$1	1/588

Suppose you spend \$1 to buy a bus pass that lets you go to 25 different restaurants in the chain and pick up entry forms. Find your expected value.

51. ***The Hog Game*** In the hog game, each player states the number of dice that he or she would like to roll. The player then rolls that many dice. If a 1 comes up on any die, the player's score is 0. Otherwise, the player's score is the sum of the numbers rolled.*

a. Find the expected value of the player's score when the player rolls one die.

b. Find the expected value of the player's score when the player rolls two dice.

c. Verify that the expected nonzero score of a single die is 4, so that if a player rolls n dice that do not result in a score of 0, the expected score is $4n$.

d. Verify that if a player rolls n dice, there are 5^n possible ways to get a nonzero score, and 6^n possible ways to roll the dice. Explain why the expected value, E, of the player's score when the player rolls n dice is then

$$E = \frac{5^n(4n)}{6^n}.$$

52. ***Football*** After a team scores a touchdown, it can either attempt to kick an extra point or attempt a two-point conversion. During the 1999–2000 NFL season, two-point conversions were successful 37% of the time and the extra-point kicks were successful 94% of the time.†

a. Calculate the expected value of each strategy.

b. Which strategy, over the long run, will maximize the number of points scored?

c. Using this information, should a team always only use one strategy? Explain.

53. ***Baseball*** The 2002 National League batting champion was Barry Bonds, with an average of .370.‡ This can be interpreted as a probability of .370 of getting a hit whenever he bats. Assume that each time at bat is an independent event. Suppose he goes to bat four times in a game.

a. Find the probability distribution for the number of hits.

b. What is the expected number of hits that Barry Bonds gets in a game?

*Bohan, James and John Shultz, "Revisiting and Extending the Hog Game," *Mathematics Teacher,* Vol. 89, No. 9, Dec. 1996, pp. 728–733.

†Leonhardt, David, "In Football, 6 + 2 Often Equals 6," *The New York Times,* Sunday, Jan. 16, 2000, pp. 4–2.

‡*The World Almanac and Book of Facts 2003,* p. 889.

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CHAPTER SUMMARY

In this chapter we continued our study of probability by introducing some elementary principles of counting. Permutations were introduced to efficiently count the number of items that can occur when grouping nonrepetitive items such that order matters. Combinations were then introduced to help us determine the number of occurrences of nonrepetitive items when order does not matter. These concepts were then used to calculate various probabilities. In particular, binomial probability was defined and applied to a variety of situations. Finally, definitions of probability distribution and expected value were given. In the next chapter we will further our study of probability by introducing the field of statistics.

KEY TERMS

8.1 **multiplication principle**
factorial notation
permutations
distinguishable permutations

8.2 **combinations**

8.4 **Bernoulli trials**
binomial experiment
binomial probability
Pascal's triangle

8.5 **random variable**
probability distribution
probability function
histogram
expected value
fair game

CHAPTER 8 REVIEW EXERCISES

1. In how many ways can 6 shuttle vans line up at the airport?

2. How many variations in first-, second-, and third-place finishes are possible in a 100-yd dash with 6 runners?

3. In how many ways can a sample of 3 oranges be taken from a bag of a dozen oranges?

4. If 2 of the oranges in Exercise 3 are rotten, in how many ways can the sample of 3 include

a. 1 rotten orange?

b. 2 rotten oranges?

c. no rotten oranges?

d. at most 2 rotten oranges?

5. In how many ways can 2 pictures, selected from a group of 5 different pictures, be arranged in a row on a wall?

6. In how many ways can the 5 pictures in Exercise 5 be arranged in a row if a certain one must be first?

7. In how many ways can the 5 pictures in Exercise 5 be arranged if 2 are landscapes and 3 are puppies and if

a. like types must be kept together?

b. landscapes and puppies are alternated?

8. In a Chinese restaurant the menu lists 8 items in column A and 6 items in column B.

a. To order a dinner, the diner is told to select 3 items from column A and 2 from column B. How many dinners are possible?

b. How many dinners are possible if the diner can select up to 3 from column A and up to 2 from column B? Assume at least one item must be included from either A or B.

9. A representative is to be selected from each of 3 departments in a small college. There are 7 people in the first department, 5 in the second department, and 4 in the third department.

a. How many different groups of 3 representatives are possible?

b. How many groups are possible if any number (at least 1) up to 3 representatives can form a group? (Each department is still restricted to at most one representative.)

10. Explain under what circumstances a permutation should be used in a probability problem, and under what circumstances a combination should be used.

11. Discuss under what circumstances the binomial probability formula should be used in a probability problem.

A basket contains 4 *black,* 2 *blue, and* 5 *green balls. A sample of* 3 *balls is drawn. Find the probabilities that the sample contains the following.*

12. All black balls

13. All blue balls

14. 2 black balls and 1 green ball

15. Exactly 2 black balls

16. Exactly 1 blue ball

17. 2 green balls and 1 blue ball

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Suppose a family plans 6 *children, and the probability that a particular child is a girl is* 1/2. *Find the probabilities that the* 6*-child family has the following children.*

18. Exactly 3 girls **19.** All girls **20.** At least 4 girls **21.** No more than 2 boys

Suppose 2 *cards are drawn without replacement from an ordinary deck of* 52. *Find the probabilities of the following results.*

22. Both cards are red. **23.** Both cards are spades. **24.** At least 1 card is a spade.

25. One is a face card and the other is not.

26. At least one is a face card.

27. At most one is a queen.

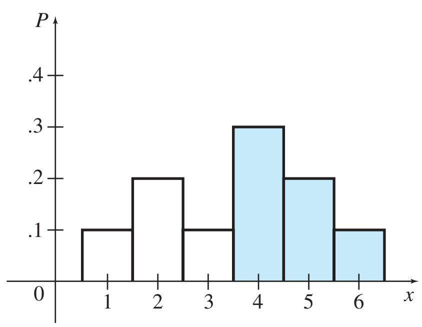
In Exercises 28 and 29, **(a)** *give a probability distribution,* **(b)** *sketch its histogram, and* **(c)** *find the expected value.*

28. A coin is tossed 3 times and the number of heads is recorded.

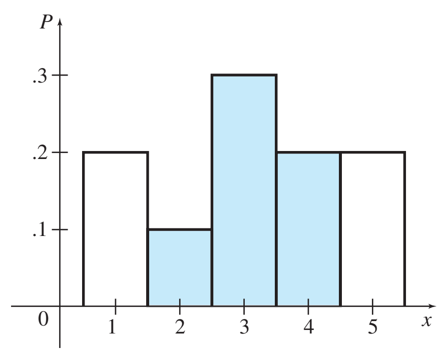
29. A pair of dice is rolled and the sum of the results for each roll is recorded.

In Exercises 30 and 31, give the probability that corresponds to the shaded region of each histogram.

30.



31.



32. You pay \$6 to play in a game where you will roll a die, with payoffs as follows: \$8 for a 6, \$7 for a 5, and \$4 for any other results. What are your expected winnings? Is the game fair?

33. Find the expected number of girls in a family of 5 children.

34. Three cards are drawn from a standard deck of 52 cards.

a. What is the expected number of aces? **b.** What is the expected number of clubs?

35. Suppose someone offers to pay you \$100 if you draw 3 cards from a standard deck of 52 cards and all the cards are clubs. What should you pay for the chance to win if it is a fair game?

36. Six students will decide which of them are on a committee by flipping a coin. Each student flips the coin, and is on the committee if he or she gets a head. What is the probability that someone is on the committee, but not all 6 students?

37. In this exercise we study the connection between sets (from Chapter 7) and combinations (from Chapter 8).

a. Given a set with n elements, what is the number of subsets of size 0? of size 1? of size 2? of size n?

b. Using your answer from part a, give an expression for the total number of subsets of a set with n elements.

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c. Using your answer from part b and a result from Chapter 7, explain why the following equation must be true:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

d. Verify the equation in part c for $n = 4$ and $n = 5$.

e. Explain what the equation in part c tells you about Pascal's triangle.

*In the following exercise, find the digit (0 through 9) that belongs in each box. This exercise is from the 1990 University Entrance Center Examination, given in Japan to all applicants for public universities.**

38. The numbers 1 through 9 are written individually on nine cards. Choose three cards from the nine, letting x, y, and z denote the numbers of the cards arranged in increasing order.

a. There are $\square\square$ such x, y, and z combinations.

b. The probability of having x, y and z all even is $\dfrac{\square}{\square\square}$.

c. The probability of having x, y and z be consecutive numbers is $\dfrac{\square}{\square\square}$.

d. The probability of having $x = 4$ is $\dfrac{\square}{\square\square}$.

e. Possible values of x range from $\square$ to $\square$. If k is an integer such that $\square \le k \le \square$, the probability that $x = k$ is $\dfrac{(\square - k)(\square - k)}{\square\square\square}$. The expected value of x is $\dfrac{\square}{\square}$.

Applications

BUSINESS AND ECONOMICS

Quality Control *A certain machine that is used to manufacture screws produces a defect rate of .01. A random sample of 20 screws is selected. Find the probabilities that the sample contains the following.*

39. Exactly 4 defective screws

40. Exactly 3 defective screws

41. No more than 4 defective screws

42. Set up the probability that the sample has 12 or more defective screws. (Do not evaluate.)

43. ***Land Development*** A developer can buy a piece of property that will produce a profit of \$16,000 with probability .7, or a loss of \$9000 with probability .3. What is the expected profit?

Exercises 44 and 45 are taken from actuarial examinations given by the Society of Actuaries.†

44. ***Product Success*** A company is considering the introduction of a new product that is believed to have probability .5 of being successful and probability .5 of being unsuccessful. Successful products pass quality control 80% of the time. Unsuccessful products pass quality control 25% of the time. If the product is successful, the net profit to the company will be \$40 million; if unsuccessful, the net loss will be \$15 million. Determine the expected net profit if the product passes quality control.

a. \$23 million **b.** \$24 million **c.** \$25 million

d. \$26 million **e.** \$27 million

45. ***Sampling Fruit*** A merchant buys boxes of fruit from a grower and sells them. Each box of fruit is either Good or

*"Japanese University Entrance Examination Problems in Mathematics," by Ling-Erl Eileen T. Wu, ed., Mathematical Association of America, 1993, p. 5. Copyright © 1993 from Wu's Japanese University Entrance Examination Problems in Mathematics, published by Mathematical Association of America.

†Course 130 Examination, Operations Research, Nov. 1989. Reprinted by permission of the Society of Actuaries.

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Bad. A Good box contains 80% excellent fruit and will earn \$200 profit on the retail market. A Bad box contains 30% excellent fruit and will produce a loss of \$1000. The a priori probability of receiving a Good box of fruit is .9. Before the merchant decides to put the box on the market, he can sample one piece of fruit to test whether it is excellent. Based on that sample, he has the option of rejecting the box without paying for it. Determine the expected value of the right to sample. (*Hint:* If the merchant samples the fruit, what are the probabilities of accepting a Good box, accepting a Bad box, and not accepting the box? What are these probabilities if he does not sample the fruit?)

a. 0 **b.** \$16 **c.** \$34 **d.** \$72 **e.** \$80

46. ***Overbooking Flights*** The March 1982 issue of *Mathematics Teacher* included "Overbooking Airline Flights," an article by Joe Dan Austin. In this article, Austin developed a model for the expected income for an airline flight. With appropriate assumptions, the probability that exactly x of n people with reservations show up at the airport to buy a ticket is given by the binomial probability formula. Assume the following: 6 reservations have been accepted for 3 seats, $p = .6$ is the probability that a person with a reservation will show up, a ticket costs \$100, and the airline must pay \$100 to anyone with a reservation who does not get a ticket. Complete the following table.

Number Who Show Up (x)	0	1	2	3	4	5	6
Airline's Income							
$P(x)$							

a. Use the table to find $E(I)$, the expected airline income from the 3 seats.

b. Find $E(I)$ for $n = 3$, $n = 4$, and $n = 5$. Compare these answers with $E(I)$ for $n = 6$. For these values of n, how many reservations should the airline book for the 3 seats in order to maximize the expected revenue?

LIFE SCIENCES

47. ***Pharmacology*** In placebo-controlled trials of Prozac®, a drug that is prescribed to fight depression, 23% of the patients who were taking the drug experienced nausea, whereas 10% of the patients who were taking the placebo experienced nausea.*

a. If 50 patients who are taking Prozac® are selected, what is the probability that 10 or more will experience nausea?

b. Of the 50 patients in part a, what is the expected number of patients who will experience nausea?

c. If a second group of 50 patients receives a placebo, what is the probability that 10 or fewer will experience nausea?

d. If a patient from a study of 1000 people, who are equally divided into two groups (those taking a placebo and those taking Prozac®), is experiencing nausea, what is the probability that he/she is taking Prozac®?

e. Since .23 is more than twice as large as .10, do you think that people who take Prozac® are more likely to experience nausea than those who take a placebo? Explain.

SOCIAL SCIENCES

48. ***Education*** In Exercise 38 of Section 8.3, we saw that a school in Bangkok requires that students take an entrance examination. After the examination, 5 students are randomly drawn from each group of 40 for automatic acceptance into the school regardless of their performance on the examination. The drawing consists of placing 35 red and 5 green pieces of paper into a box. If the lottery is changed so that each student picks a piece of paper from the box and then returns the piece of paper to the box, find the probability that exactly 5 of the 40 students will choose a green piece of paper.†

GENERAL INTEREST

In Exercises 49–52, **(a)** *give a probability distribution,* **(b)** *sketch its histogram, and* **(c)** *find the expected value.*

49. ***Candy*** According to officials of Mars, the makers of M&M Plain Chocolate Candies, 20% of the candies in each bag are red.‡ Four candies are selected from a bag and the number of red candies is recorded.

50. ***Women Athletes*** In 1992, the Big 10 collegiate sports conference moved to have women compose at least 40% of its athletes within 5 years.§ Suppose they exactly achieve the 40% figure, and that 5 athletes are picked at random from Big 10 universities. The number of women is recorded.

*Advertisement in *The New England Journal of Medicine,* Vol. 338, No. 9, Feb. 26, 1998, for Prozac®, developed and marketed by Eli Lily and Company.

†"Media Clips," *Mathematics Teacher,* Vol. 92, No. 8, No. 1999. Copyright 1999. Used with permission from the National Council of Teachers of Mathematics. All rights reserved.

‡http://global.mms.com/us/about/products/milkchocolate.jsp.

§*Chicago Tribune,* Apr. 28, 1993, p. 19.

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51. ***Race*** In the mathematics honors society at a college, 2 of the 8 members are African American. Three members are selected at random to be interviewed by the student newspaper, and the number of African Americans is noted.

52. ***Homework*** In a small class of 10 students, 3 did not do their homework. The professor selects half of the class to present solutions to homework problems on the board, and records how many of those selected did not do their homework.

53. ***Lottery*** A lottery has a first prize of \$5000, two second prizes of \$1000 each, and two \$100 third prizes. A total of 10,000 tickets is sold, at \$1 each. Find the expected winnings of a person buying 1 ticket.

54. ***Contests*** At one time, game boards for a United Airlines contest could be obtained by sending a self-addressed, stamped envelope to a certain address. The prize was a ticket for any city to which United flies. Assume that the value of the ticket was \$2000 (we might as well go first-class), and that the probability that a particular game board would win was $1/8000$. If the stamps to enter the contest cost 37¢ and envelopes cost 4¢ each, find the expected winnings for a person ordering 1 game board. (Notice that 2 stamps and envelopes were required to enter.)

55. ***Lottery*** On June 23, 2003, an interesting thing happened in the Pennsylvania Lottery's Big 4, in which a four-digit number from 0000 to 9999 is chosen twice a day.* On this day, the number 3199 was chosen both times.

a. What is the probability of the same number being chosen twice in one day?

b. What is the probability of the number 3199 being chosen twice in one day?

56. ***Lottery*** In the Pennsylvania Lottery's Daily Number game, a three-digit number between 000 and 999 is chosen each day.† The favorite number among players is 000, which on July 28, 2003, was the winning number for the tenth time since 1977. Find the number of times that 000 would be expected to win in 26 years of play. (Assume that the game is played 365 days a year, ignoring leap years and the fact that before 1990, the game was not played on Christmas or New Year's Day. Also ignore the fact that since February 2003, the game has been played twice a day.)

57. ***Lottery*** New York has a lottery game called Quick Draw, in which the player can pick anywhere from 1 up to 10 numbers from 1 to 80. The computer then picks 20 numbers, and how much you win is based on how many of your numbers match the computer's. For simplicity, we will only consider the two cases in which you pick 4 or 5 numbers. The payoffs for each dollar that you bet are given in the table below.

	How Many Numbers Match the Computer's Numbers					
	0	**1**	**2**	**3**	**4**	**5**
You Pick 4	0	0	1	5	55	
You Pick 5	0	0	0	2	20	300

a. According to the Quick Draw playing card, the "Overall Chances of Winning" when you pick 4 are "1:3.86," while the chances when you pick 5 are "1:10.34." Verify these figures.

b. Find the expected value when you pick 4 and when you pick 5, betting \$1 each time.

c. Based on your results from parts a and b, are you better off picking 4 numbers or picking 5? Explain your reasoning.

58. ***Murphy's Law*** Robert Matthews wrote an article about Murphy's Law, which says that if something can go wrong, it will.‡ He considers Murphy's Law of Odd Socks, which says that if an odd sock can be created it will be, in a drawer of 10 loose pairs of socks.

a. Find the probability of getting a matching pair when the following numbers of socks are selected at random from the drawer.

i. 5 socks **ii.** 6 socks

b. Matthews says that it is necessary to rummage through 30% of the socks to get a matching pair. Using your answers from part a, explain precisely what he means by that.

c. Matthews claims that if you lose 6 socks at random from the drawer, then it is 100 times more likely that you will be left with the worst possible outcome—6 odd socks—than with a drawer free of odd socks. Verify this calculation by finding the probability that you will be left with 6 odd socks and the probability that you will have a drawer free of odd socks.

57. ***Baseball*** The number of runs scored in 16,456 half-innings of the 1986 National League Baseball season was

*http://www.palottery.com.
†Ibid.
‡Matthews, Robert, "Why Does Toast Always Land Butter-Side Down?" *Sunday Telegraph,* March 17, 1996, p. 4.

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analyzed by Hal Stern. Use the following table to answer the following questions.*

a. What is the probability that a given team scored 5 or more runs in any given half-inning during the 1986 season?

b. What is the probability that a given team scored fewer than 2 runs in any given half-inning of the 1986 season?

c. What is the expected number of runs that a team scored during any given half-inning of the 1986 season? Interpret this number.

Runs	Frequency	Probability
0	12,087	.7345
1	2451	.1489
2	1075	.0653
3	504	.0306
4	225	.0137
5	66	.0040
6	29	.0018
7	12	.0007
8	5	.0003
9	2	.0001

*J. Laurie Snell's report of Hal Stern's analysis in *Chance News 7.05,* Apr. 27–May 26, 1998.

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EXTENDED APPLICATION: Optimal Inventory for a Service Truck

For many different items it is difficult or impossible to take the item to a central repair facility when service is required. Washing machines, large television sets, office copiers, and computers are only a few examples of such items. Service for items of this type is commonly performed by sending a repair person to the item, with the person driving to the location in a truck containing various parts that might be required in repairing the item. Ideally, the truck should contain all the parts that might be required. However, most parts would be needed only infrequently, so that inventory costs for the parts would be high.

An optimum policy for deciding on which parts to stock on a truck would require that the probability of not being able to repair an item without a trip back to the warehouse for needed parts be as low as possible, consistent with minimum inventory costs. An analysis similar to the one below was developed at the Xerox Corporation.*

To set up a mathematical model for deciding on the optimum truck-stocking policy, let us assume that a broken machine might require one of 5 different parts (we could assume any number of different parts—we use 5 to simplify the notation). Suppose also that the probability that a particular machine requires part 1 is p_1; that it requires part 2 is p_2; and so on. Assume also that failures of different part types are independent, and that at most one part of each type is used on a given job.

Suppose that, on the average, a repair person makes N service calls per time period. If the repair person is unable to make a repair because at least one of the parts is unavailable, there is a penalty cost, L, corresponding to wasted time for the repair person, an extra trip to the parts depot, customer unhappiness, and so on. For each of the parts carried on the truck, an average inventory cost is incurred. Let H_i be the average inventory cost for part i, where $1 \le i \le 5$.

Let M_1 represent a policy of carrying only part 1 on the repair truck, M_{24} represent a policy of carrying only parts 2 and 4, with M_{12345} and M_0 representing policies of carrying all parts and no parts, respectively.

For policy M_{35}, carrying parts 3 and 5 only, the expected cost per time period per repair person, written $C(M_{35})$, is

$$C(M_{35}) = (H_3 + H_5) + NL[1 - (1 - p_1)(1 - p_2)(1 - p_4)].$$

(The expression in brackets represents the probability of needing at least one of the parts not carried, 1, 2, or 4 here.) As further examples,

$$C(M_{125}) = (H_1 + H_2 + H_5) + NL[1 - (1 - p_3)(1 - p_4)],$$

while

$$\begin{aligned} C(M_{12345}) &= (H_1 + H_2 + H_3 + H_4 + H_5) + NL[1 - 1] \\ &= H_1 + H_2 + H_3 + H_4 + H_5, \end{aligned}$$

and

$$C(M_0) = NL[1 - (1 - p_1)(1 - p_2)(1 - p_3)(1 - p_4)(1 - p_5)].$$

To find the best policy, evaluate $C(M_0)$, $C(M_1), \ldots, C(M_{12345})$, and choose the smallest result. (A general solution method is in the *Management Science* paper.)

Example

Suppose that for a particular item, only 3 possible parts might need to be replaced. By studying past records of failures of the item, and finding necessary inventory costs, suppose that the following values have been found.

p_1	p_2	p_3
.09	.24	.17

H_1	H_2	H_3
$15	$40	$9

*Smith, Stephen, John Chambers, and Eli Shlifer, "Optimal Inventories Based on Job Completion Rate for Repairs Requiring Multiple Items," *Management Science*, Vol. 26, No. 8, Aug. 1980. © 1980 by The Institute of Management Sciences.

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Suppose $N = 3$ and L is \$54. Then, as an example,

$$\begin{aligned} C(M_1) &= H_1 + NL[1 - (1 - p_2)(1 - p_3)] \\ &= 15 + 3(54)[1 - (1 - .24)(1 - .17)] \\ &= 15 + 3(54)[1 - (.76)(.83)] \\ &\approx 15 + 59.81 = 74.81. \end{aligned}$$

Thus, if policy M_1 is followed (carrying only part 1 on the truck), the expected cost per repair person per time period is \$74.81. Also,

$$\begin{aligned} C(M_{23}) &= H_2 + H_3 + NL[1 - (1 - p_1)] \\ &= 40 + 9 + 3(54)(.09) = 63.58, \end{aligned}$$

so that M_{23} is a better policy than M_1. By finding the expected values for all other possible policies (see the exercises), the optimum policy may be chosen.

Exercises

1. Refer to the example and find the following.

a. $C(M_0)$ **b.** $C(M_2)$ **c.** $C(M_3)$ **d.** $C(M_{12})$
e. $C(M_{13})$ **f.** $C(M_{123})$

2. Which policy leads to the lowest expected cost?

3. In the example, $p_1 + p_2 + p_3 = .09 + .24 + .17 = .50$. Why is it not necessary that the probabilities add up to 1?

4. Suppose an item to be repaired might need one of n different parts. How many different policies would then need to be evaluated?

Directions for Group Project

Suppose you and three others are employed as service repair persons and that you have some disagreement with your supervisor as to the quantity and type of parts to have on hand for your service calls. Use the answers to Exercises 1–4 to prepare a report with a recommendation to your boss on optimal inventory. Make sure that you describe each concept well since your boss is not mathematically minded.

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