Define the unit ramp function by

$$H(t)=\begin{cases}0 & t<0\\ t & t>0\end{cases}$$

1. Determine the Laplace transform of H(t)
2. Use the Laplace transform to solve the ODE

$$y''+y=H(t-1) \qquad y(0)=0 \quad y'(0)=0 \qquad (*)$$

Questions:

1. To determine the Laplace transform, do we use

   $L(H(t))=\int_0^\infty e^{-st}H(t)dt$ or $L(H(t))=\int_{-\infty}^\infty e^{-st}H(t)dt$ (note the difference in integration limits). I know both integrals will give the same answer. But I am confused because H(t) is also defined for t<0 and therefore it seems a bit unnatural to let the lower limit of integration be 0.

   In any case, the answer to this part is $H(s)=\dfrac{1}{s^2}$

2. Taking the Laplace transform of the given equation yields

   $$s^2Y(s)-sy(0)-y'(0)+sY(s)-y(0)=\frac{e^{-s}}{s^2}\Rightarrow Y(s)=\frac{e^{-s}}{s^3(s+1)}$$

   Thus to find the solution y(t) we must determine the inverse Laplace transformation of the above. This is where I am having trouble since rewriting the expression for Y(s) seems to give different expressions for y(t), non of which actually satisfies (*) when substituted. A few of these options are given here:

   CASE 1:

   $$Y(s)=\frac{e^{-s}}{s^3(s+1)}=\frac{e^{-s}}{s^2}\left(\frac{1}{s(s+1)}\right)=\frac{e^{-s}}{s^2}\left(\frac{1}{s}-\frac{1}{s+1}\right)\Rightarrow y(t)=H(t-1)\cdot(-e^{-t}+1)$$

   CASE 2:

   $$Y(s)=\frac{e^{-s}}{s^3(s+1)}=\frac{e^{-s}}{s}\left(\frac{1}{s^2(s+1)}\right)=\frac{e^{-s}}{s}\left(-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s+1}\right)\Rightarrow y(t)=u(t-1)\cdot(-1+t+e^{-t})$$

   where $u(t)$ denotes the Heaviside function. Alternatively, we know from part 1 that the Laplace transform of H(s)=1/s^2 so we could also have written

   $$y(t)=u(t-1)\cdot(-1+H(t)+e^{-t})$$

CASE 3:

$$Y(s)=\frac{e^{-s}}{s^3(s+1)}=e^{-s}\cdot\frac{1}{s^3(s+1)}=e^{-s}\left[\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^3}-\frac{1}{s+1}\right]\Rightarrow y(t)=\left(1-t+t^2-e^{-t}\right)u(t-1)$$

None of these solutions actually seems to give (*) if we calculate the second derivative of y(t) and substitute it into (*), the RHS does not turn out to be H(t-1)

My question therefore is:

1. What is the correct Laplace transform in this case.
2. Please show step by step how to obtain this (i.e. if we have to use some method like using residues, please show how to do this in detail, because I don't really know how to do this.

p.s. Maple seems to give even more different Inverse Laplace transforms, but I am not really interested in how to do it with Maple, since I want to know how to do it by hand.