NOTE The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither (as in Example 5). This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

EXAMPLE 6 Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

SOLUTION You can use the differentiation rules to check that the first two derivatives are

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, and 6.

▪ Try reproducing the graph in Figure 11 with a graphing calculator or computer. Some machines produce the complete graph, some produce only the portion to the right of the y-axis, and some produce only the portion between $x = 0$ and $x = 6$. An equivalent expression that gives the correct graph is

$$y = (x^2)^{1/3} \cdot \frac{6 - x}{|6 - x|}|6 - x|^{1/3}$$

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	+	−	+	−	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	−	+	+	−	decreasing on $(4, 6)$
$x > 6$	−	+	+	−	decreasing on $(6, \infty)$

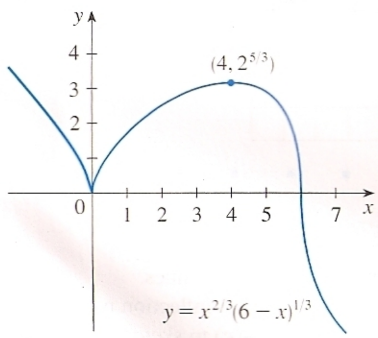

FIGURE 11

To find the local extreme values we use the First Derivative Test. Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum. Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum. The sign of f' does not change at 6, so there is no minimum or maximum there. (The Second Derivative Test could be used at 4 but not at 0 or 6 since f'' does not exist at either of these numbers.)

Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geqslant 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$ and $f''(x) > 0$ for $x > 6$. So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ and concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$. The graph is sketched in Figure 11. Note that the curve has vertical tangents at $(0, 0)$ and $(6, 0)$ because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$. ■

4.3 EXERCISES

1–8 ▪
(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

1. $f(x) = x^3 - 12x + 1$
2. $f(x) = x^4 - 4x - 1$
3. $f(x) = x - 2\sin x, \quad 0 < x < 3\pi$
4. $f(x) = \dfrac{x^2}{x^2 + 3}$
5. $f(x) = xe^x$
6. $f(x) = x^2e^x$
7. $f(x) = (\ln x)/\sqrt{x}$
8. $f(x) = x \ln x$

9–10 ▪ Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

9. $f(x) = x + \sqrt{1 - x}$
10. $f(x) = \dfrac{x}{x^2 + 4}$

11. Suppose f'' is continuous on $(-\infty, \infty)$.
    (a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f?
    (b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

12. (a) Find the critical numbers of $f(x) = x^4(x - 1)^3$.
    (b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
    (c) What does the First Derivative Test tell you?