where

$$\sum_{i=1}^{r} \alpha_i = 1, \quad \alpha_i \geq 0, \quad i = 1, \ldots, r$$

with the additional restriction that at most two adjacent α_i terms are nonzero.

For a given $\hat{x}_j$, $g_j(\hat{x}_j)$ as determined by Equation (7) is a point on one of the line segments if no more than two of the α_i terms are positive and adjacent—i.e., of the form α_i, α_{i+1}. This condition can be achieved with the additional constraints

$$\begin{aligned} &\alpha_1 \leq y_1 \\ &\alpha_i \leq y_{i-1} + y_i, \ \ i = 2, \ldots, r-1 \\ &\alpha_r \leq y_{r-1} \\ &\sum_{i=1}^{r-1} y_i = 1 \\ &y_i = 0 \text{ or } 1, \ \ i = 1, \ldots, r-1 \end{aligned}$$

From these relationships, it follows that for some q, $1 \leq q \leq r-1$, $y_q = 1$, and $y_i = 0$, $i \neq q$. Thus, $\alpha_q \leq 1$, $\alpha_{q+1} \leq 1$, and $\alpha_i = 0$ for all $i \neq q$ and $i \neq q + 1$.

EXERCISES

1. Consider the IP model

$$\begin{aligned} \text{Maximize } & 2x_1 + 5x_2 \\ \text{subject to } & x_1 + x_2 \leq 15 \\ & -x_1 + x_2 \leq 2 \\ & x_1 - x_2 \geq 2 \\ & x_1 + x_2 \geq 2 \\ & x_1, x_2 \geq 0 \text{ and integer} \end{aligned}$$

The following problems are cumulative in that each part is based on the answer(s) to the previous part(s).

(a) Rewrite the model using only binary variables.

(b) Rewrite the model as a minimization problem with all "less than or equal to" constraints.

(c) Rewrite the model as a minimization problem with all positive objective function coefficients.

2. A company is considering three major research projects labeled A, B, and C. Each of the projects can be selected for any of the next 3 years or may be omitted from the portfolio entirely. The total return for each project based on the year it is selected is given in the table. This return captures all relevant cash flows, including investments. It also includes the effects of the time value of money. The investment required for each project occurs entirely within the year for which it is selected. The goal is to maximize the total return. This problem has the following constraints. No more than two projects can be selected in any year. Total investment in any year cannot exceed 9. Project B must be selected after project A. Project A and B cannot be selected in the same year. A project can be selected at most once. Formulate and solve the problem as an IP. Define all notation.

	Total return for project		
Year	A	B	C
1	7	6	4
2	5	4	4
3	8	7	5
Investment	5	3	5

3. A computer service company needs to establish communications among five cities. An analysis of various media has determined that the monthly cost of connecting a pair of cities i and j with a link is c_{ij}, as shown in the matrix. The connection allows communications in both directions. The cost of establishing interconnection facilities at each city depends on the number of links incident to the city. Note that these are node costs rather than arc costs.

City	1	2	3	4	5
1	—	15	13	19	21
2	15	—	10	24	14
3	13	10	—	14	17
4	19	24	14	—	12
5	21	14	17	12	—

- If one link touches a city, the cost is d_1
- The second link touching the city adds the cost d_2
- The third link touching the city adds the cost d_3
- These costs are related as follows: $d_1 > d_2 > d_3$

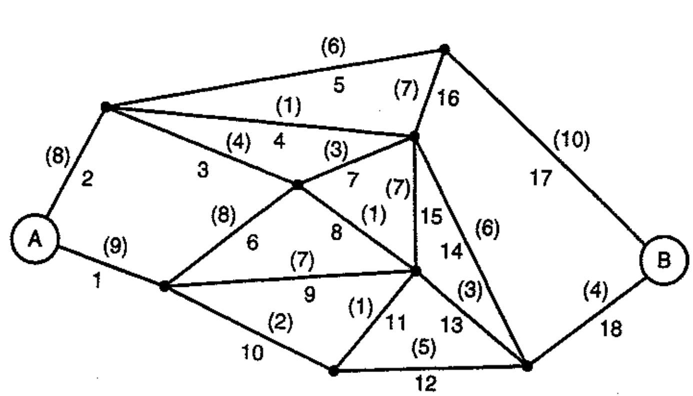
Formulate and solve a 0-1 ILP model that incorporates the following information.

- The objective is to minimize monthly cost
- Each city must be touched by at least one connection
- The links selected must form a tree
- No more than three links can touch any one city
- $d_1 = 5$, $d_2 = 3$, $d_3 = 1$

4. The figure shows a road network between two cities A and B located in different states. The federal government wants to place inspection stations on the roads so that all traffic moving between the cities must pass through at least one station. The cost of establishing a station on road k is c_k, as indicated by the numbers in parentheses in the figure.

   (a) Show that the problem of selecting the minimum cost locations of stations can be modeled as a set covering problem. (*Hint*: the rows of the **A** matrix will represent paths between the two cities, and the columns will represent individual roads.)

   (b) Describe a more efficient way to solve this problem using one of the standard network models.



Road network between cities A and B

5. (*Symmetric TSP*) Consider an undirected graph with m edges and n nodes. Develop an ILP model for the symmetric traveling salesman problem. In the model, let x_e be a binary variable equal to 1 if edge e is used, and 0 otherwise. Also, let S be a proper subset of the node set N, let $E(S)$ be the set

of edges whose two endpoints are contained in S, and let $\delta(j)$ be the set of edges incident to node j. The cost of traversing edge e is c_e.

6. (*Prize collecting TSP*) A variant of the traveling salesman problem occurs when the salesman receives a profit of f_j for visiting city $j \in N$. His tour must start at city 1 and include at least two other cities. A cost of c_e is incurred if he traverses edge e. Unlike the traditional TSP, however, the salesman does not have to visit all n cities. The goal is to find a tour that maximizes the difference between profits and travel costs subject to these restrictions. Formulate an ILP that can be used to solve this problem. Use the notation introduced in the preceding exercise as well as any new notation deemed necessary.

7. It is possible to replace the subtour elimination constraint C3 in the text with the following constraint:

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1, \quad S \subset N, S \neq \varnothing \qquad \text{(SEC)}$$

To see why this is true, show that C3 is a linear combination of the assignment constraints C1 and C2, and (SEC). C1, C2, and C3 are from the TSP discussion in Section 7.6.

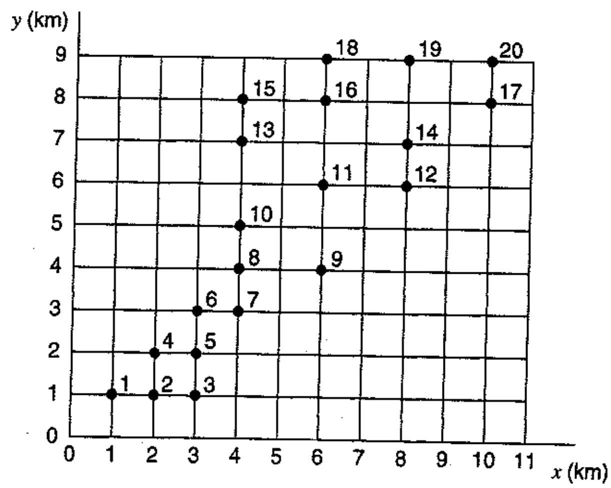
8. Give a general IP formulation for the cutting stock problem described in Section 7.7.

9. For the cutting stock example in Section 7.7, write the objective function for each of the following two cases. Solve each case.

   (a) The goal is to find the number of each pattern so that the excess that must be discarded is minimized

   (b) In practice, each pattern requires a certain number of cuts. For instance, pattern 1 requires only one cut whereas pattern 2 requires two cuts. The goal is to find the selection of patterns that minimizes the total number of cuts

10. The figure shows well targets identified by geologists in an offshore oil field. The wells are to be drilled from platforms that are very expensive to erect. To reduce the cost, several wells may be drilled from the same platform. The cost of a platform designed to drill k wells is c_k. To drill a set of wells, the platform is located at the centroid of the wells and directional drilling is used to reach each target. The operational cost of drilling a well is $a(h + d)^2$, where h is the horizontal distance from the platform location to the target, d is the depth, and a is a constant. The depth is assumed to be constant over the field.



Well targets for offshore drilling

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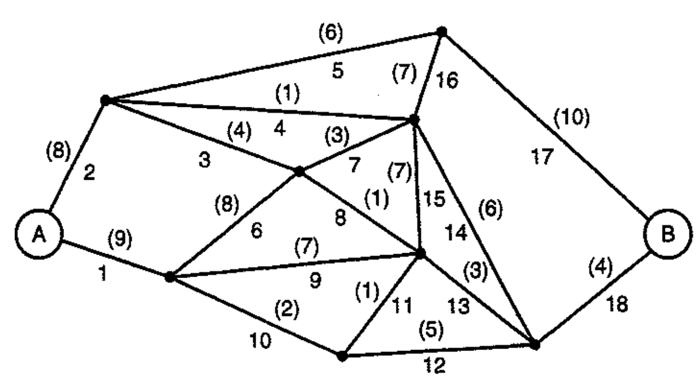

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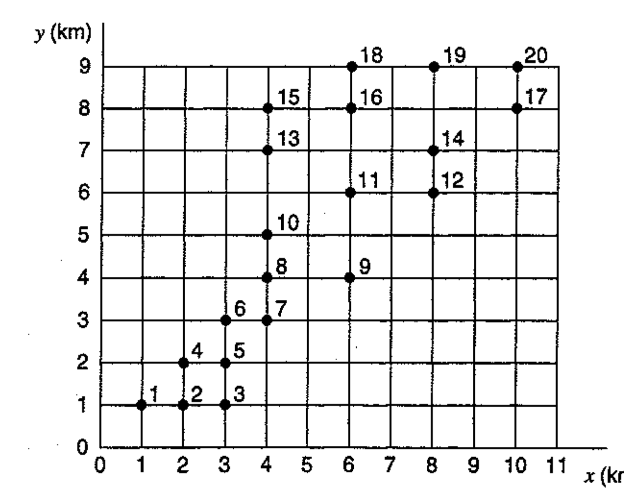

Well targets for offshore drilling