EXERCISES

1. Consider the following linear program.

$$\begin{aligned} \text{Maximize } z = \; & 5x_1 + 3x_2 \\ \text{subject to } \; & 3x_1 + 5x_2 \le 15 \\ & 5x_1 + 2x_2 \le 10 \\ & -x_1 + x_2 \le 2 \\ & x_2 \le 2.5 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

(a) Show the equality form of the model.
(b) Sketch the graph of the feasible region and identify the extreme point solutions. From this representation, find the optimal solution.
(c) Analytically determine all solutions that derive from the intersection of two constraints or nonnegativity restrictions. Identify whether or not these solutions are feasible, and indicate the corresponding objective function values. Which one is optimal?
(d) Let the slack variables for the first two constraints, call them x_3 and x_4, be the axes of the graph, and sketch the geometric representation of the model. Show an objective isovalve contour in these variables, and from it determine the optimal solution.

2. You are given the following linear program.

$$\begin{aligned} \text{Maximize } z = \; & 8x_1 + 4x_2 + 7x_3 - 3x_4 \\ \text{subject to } \; & 2x_1 - 2x_2 + 3x_3 - 4x_4 = 12 \\ & 3x_1 + 8x_2 - x_3 + 7x_4 = 18 \\ & x_j \ge 0, \; j = 1, \ldots, 4 \end{aligned}$$

(a) Select x_3 and x_4 as the axes and sketch the feasible region and an objective isovalue contour in terms of these variables. Identify the extreme points of the region.
(b) Analytically determine the set of all solutions that are intersections of two constraints or nonnegativity restrictions. Identify whether or not the solutions are feasible, and from the feasible subset select the optimal solution by evaluating the objective function at each one.

3. Consider the linear program

$$\begin{aligned} \text{Maximize } z = \; & 10x_1 + 5x_2 + 8x_3 - 3x_4 \\ \text{subject to } \; & -2x_1 + x_2 + 2x_3 - 3x_4 \ge 12 \\ & x_1 + x_2 + x_3 + x_4 \le 20 \\ & x_j \ge 0, \; j = 1, \ldots, 4 \end{aligned}$$

(a) Construct the equality form of the model by introducing slack variables x_5 and x_6 for the two constraints.
(b) What is an upper bound on the number of basic solutions?
(c) Analytically determine the set of basic solutions by listing all possible selections of the basic variables. Identify which solutions are feasible and which are infeasible. Compute the objective function values for the feasible solutions, and select the optimal solution.

4. Given the linear program

$$\begin{aligned} \text{Maximize } z = \; & x_1 + 2.5x_2 + x_3 \\ \text{subject to } \; & x_1 + x_2 \ge 10 \\ & x_2 + x_3 \ge 10 \\ & x_j \ge 0, \; j = 1, \ldots, 3 \end{aligned}$$

Algebraically determine the set of all basic solutions and identify whether or not each solution is feasible. Determine the optimal solution by evaluating the objective function at each feasible point.

5. The following tableau is not in simplex form.

Row	Basic	Coefficients						RHS
		z	x_1	x_2	x_3	x_4	x_5	
0	z	1	1	−1	−1	0	2	20
1	—	0	1	5	−1	1	12	12
2	—	0	0	8	1	2	16	16

(a) Write the set of equations described by the tableau as it stands.
(b) Using linear operations, convert the tableau into simplex form by making x_1 and x_4 basic. What is the solution corresponding to the new tableau?
(c) From the new tableau, predict the effects of increasing x_5 by 1, by 0.5, and by 2.
(d) From the new tableau, predict the effects of increasing x_3 by 1, by 0.5, and by 2.

6. Starting with the tableau found in Exercise 5(b), consider the three cases that follow. Construct the new tableau in the simplex form, write the basic solution obtained, and use the marginal information available from the tableau to comment on any characteristics that the solution exhibits. Each part of this problem refers to the original basis.

(a) Allow x_2 to replace x_4 as a basic variable.
(b) Allow x_5 to replace x_4 as a basic variable.
(c) Allow x_5 to replace x_1 as a basic variable.

7. For the example problem in Section 3.5, start with the basic solution labeled #1 in Figure 3.5. Sequentially change the basis by allowing variables to enter and leave so that the basic solutions are #2, #3, #4, and #5, in that order. Show the equations in the simplex form for each of the four cases, and sketch the view of the feasible region described for each basis.

8. This is an exercise designed to illustrate the pivoting process. In each case listed, start from the following tableau and let the specified variable enter the basis. Specify which variable should leave the basis, predict the new basic solution, and predict the new value of the objective function. Indicate whether the objective value increases or decreases.

(a) x_1 enters.
(b) x_5 enters.
(c) x_6 enters.
(d) x_7 enters.
(e) x_{10} enters.

Row	Basic	Coefficients											RHS
		z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
0	z	1	0	0	0	0	−0.2	0.2	−0.2	0	0	0.6	11
1	x_3	0	2	0	1	0	1	0	−1	0	0	2	5
2	x_4	0	−2	0	0	1	2	1	2	0	0	−4	0
3	x_8	0	0	0	0	0	0	1	1	1	0	−3	12
4	x_9	0	0	0	0	0	0	2	1	0	1	−9	9
5	x_2	0	1	1	0	0	1	−1	−1	0	0	3	3

9. The following tableau has been found at an intermediate stage of the simplex algorithm for a maximization problem. In each part of this exercise, start from the tableau and perform the suggested pivot

operation. Show the complete tableau obtained and comment on any special characteristics that it exhibits. The parts are not cumulative.

		Coefficients						
Row	Basic	z	x_1	x_2	x_3	x_4	x_5	RHS
0	z	1	0	−2	2	0	−2	−2
1	x_1	0	1	1	−1	0	4	4
2	x_4	0	0	4	1	1	8	8

(a) Let x_2 enter the basis and let x_1 leave the basis.
(b) Let x_2 enter the basis and let x_4 leave the basis.
(c) Let x_3 enter the basis and let x_1 leave the basis.
(d) Let x_3 enter the basis and let x_4 leave the basis.
(e) Let x_5 enter the basis and let x_1 leave the basis.
(f) Let x_5 enter the basis and let x_4 leave the basis.

10. You are given the following tableau.

		Coefficients							
Row	Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
0	z	1	−3	0	0	−3	−2	0	10
1	x_3	0	4	0	1	5	−3	0	4
2	x_2	0	2	1	0	−3	1	0	2
3	x_6	0	1	0	0	2	−2	1	2

Starting from this tableau in each case, perform the indicated operations to derive a new tableau.

(a) Let x_1 enter the basis.
(b) Let x_4 enter the basis.
(c) Let x_5 enter the basis.
(d) Which of the three operations results in the greatest increase in the objective function? Referring to the primal simplex algorithm in Section 3.6, rewrite Step 2(a) so that the variable chosen to enter the basis is the one that produces the greatest increase in the objective function. Would you suggest that this procedure replace the steepest ascent rule? Explain.

11. Using the tableau given in Exercise 10, if x_1 enters the basis, there is a tie for the variable that leaves the basis. Starting from the original tableau, consider the two possibilities and determine what the solution will be in each case.

(a) Let x_3 leave the basis.
(b) Let x_2 leave the basis.

12. **(a)** What special property does the following tableau exhibit?

		Coefficients							
Row	Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
0	z	1	−3	2	0	0	−1	0	4
1	x_3	0	2	0	1	0	0	0	13
2	x_4	0	−2	1	0	1	−7	0	7
3	x_6	0	4	2	0	0	−2	1	2

(b) Use the most negative rule to select the entering variable. Proceed until the algorithm terminates.

13. Find all alternative optima in the following tableau.

		Coefficients							
Row	Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
0	z	1	0	0	0	0	2	4	10
1	x_3	0	4	0	1	5	–3	2	4
2	x_2	0	2	1	0	–3	1	3	2

14. Convert the following problem into simplex form and solve with the Math Programming Excel Add-in. What are the values of the original problem variables?

$$\begin{aligned} \text{Minimize } z = 3x_1 &+ 2x_2 \\ \text{subject to } \quad x_1 &- x_2 = -11 \\ -x_1 &- 2x_2 \le -10 \\ x_1 & \quad \ge -3 \\ x_1 \text{ and } x_2 & \text{ are unrestricted} \end{aligned}$$

15. Drop the second constraint from the example problem in Section 3.2 and solve it. Comment on any special characteristics of the optimal solution.

16. Add the simple lower bound constraints $x \ge -5$ and $y \ge -5$ to the example problem in Section 3.2. Solve the problem with these constraints explicitly included and comment on any special characteristics of the optimal solution.

17. Repeat Exercise 16, but use a transformation to eliminate the lower bounds on x and y from the constraint set (i.e., do not treat x and y as unrestricted variables).

18. Add the constraint $x_1 - x_2 \ge 0$ to the example problem in Section 3.8, and solve it using the two-phase method.

19. For the following problem, use the phase 1 procedure to find a feasible solution. Set up the tableau to begin phase 2.

$$\begin{aligned} \text{Maximize } z = 2x_1 - 3x_2 + x_3 - 4x_4 & \\ \text{subject to } \quad 2x_1 - x_2 + 3x_3 - 5x_4 &\le 20 \\ x_1 + 2x_2 - x_3 + 4x_4 &\ge 2 \\ x_4 &\le 20 \\ x_j \ge 0, j = 1, \dots, 4 & \end{aligned}$$

20. Consider the linear program

$$\begin{aligned} \text{Maximize } z = 2x_1 + 3x_2 + x_3 + 4x_4 & \\ \text{subject to } \quad x_1 - x_3 + x_4 &\le 5 \\ -x_1 + 2x_2 + x_4 &\le 6 \\ x_2 + 2x_3 + 0.5x_4 &\le 8 \\ x_j \ge 0, j = 1, \dots, 4 & \end{aligned}$$

After several pivots, the simplex tableau appears as follows.

		Coefficients								
Row	Basic	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
0	z	1	0	–1.5	0	–1.25	2	0	1.5	22
1	x_1	0	1	0.5	0	1.25	1	0	0.5	9
2	x_6	0	0	2.5	0	2.25	1	1	0.5	15
3	x_3	0	0	0.5	1	0.25	0	0	0.5	4