Topics covered

Probability theory:

- Elements from set theory
- Basics of probability theory

Problem 1
Consider the experiment of flipping a coin three times.

a. What is the sample space?
b. Which sample outcomes make up the event *A*: at least two of the coins show tails?

Problem 2
Use a Venn diagram to illustrate the second of the DeMorgan's Laws: $(A\cup B)^C = A^C \cap B^C$
(the complement of the union is the intersection of the individual complements).

Hint: proceed as in the example solved in class: draw a Venn diagram of the LHS and of the RHS, and show they are the same.

Problem 3

A supervisor needs to select two workers for a job (consisting of two identical tasks) out of a group of five workers. Suppose that the workers vary in competence, 1 being the best, 2 – the second best, and so on, and 5 – the worst. The supervisor does not wish to show any biases in his selection, so he decides to select the two workers at random.

(a) What are the possible outcomes of this experiment, i.e. what is the sample space?

Note: In answering this question, please, use the following notation: (k,l) meaning "the supervisor selected workers k and l". E.g. (1,3) meaning "the supervisor selected workers 1 and 3"; (4,5) meaning "the supervisor selected workers 4 and 5", etc.)

(b) How many elements are there in the sample space? (Just count them). Verify that this number is equal to the number of ways to select 2 out 5 elements where order does not matter, i.e. verify this is the same as combinations C_2^5.

(c) Now define events A as: the employer selects the best worker and one of the two poorest workers (i.e. he selects workers 1 and 4 OR workers 1 and 5).
Find the probability of A, P(A).

Hint: In part (c) you may find it easier to denote:
A1: selecting (1,4)
A2: selecting (1,5).
Then you need to find $P(A) = P(A1 \cup A2)$. This is easy as A1 and A2 are mutually exclusive.

Problem 4

Let A and B be two events defined on a sample space S such that P(A)=0.3, P(B)=0.5, and $P(A\cup B)=0.7$. Find:

a. $P(A\cap B)$

Hint: express $P(A\cap B)$ from the expression for $P(A\cup B)$.

b. $P(A^C \cup B^C)$

Hint: use De Morgan's law: $P(A^C\cup B^C)= P(A \cap B)^C$.

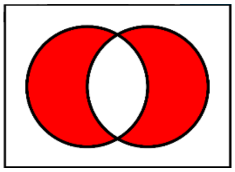
c. $P(A^C \cap B)$

Hint: express $P(A^C \cap B)$ from the total probability definition for B.

Problem 5

Let A and B be any two events defined on S. Suppose that P(A) = 0.4, P(B) = 0.5, and $P(A\cap B)=0.1$. What is the probability that <u>either A or B occur, but not both</u>?

Hint: A Venn diagram would help: we are looking for the probability of the event defined by the shaded area:



Problem 6

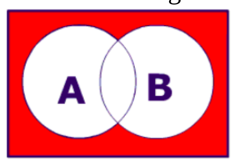
Let A and B be any two events defined on S. Suppose that P(A) = 0.4, P(B) = 0.3, and $P(A \cap B)=0.05$. What is the probability that neither of the events A and B occurs?

Hint: Most problems asking to find the probability that neither of several events are solved easiest by expressing this probability as: P(neither of them occurs) = 1 – P(at least one occurs). Or, in terms of notation, we need to find:

$P(A^C \cap B^C) = P[(A \cup B)^C]$ (using one of DeMorgan's Laws)

$= 1 - P(A \cup B)$

It is easy to see that $P(A^C \cap B^C) = 1 - P(A \cup B)$ from a Venn diagram:



So, essentially, you just need to find $P(A \cup B)$ using Theorem 4 from the lecture notes.
If you still experience difficulties, see solved example 2.3.4b. in the text.

Problem 7

Let A and B be any two events defined on S. Using the definition of independence and the axioms of probability, prove that:

a. if A and B are independent, then so are A and B^C.

Hint: We solved this example in class.

Generally, you need to follow the steps below:

- By definition of independent events, in order to show that $A \perp\!\!\!\perp B^C$, it is enough to show

that $P(A \cap B^C) = P(A)\,P(B^C)$.

- Express $P(A \cap B^C)$ from the total probability definition for A:
  $P(A) = P(A \cap B) + P(A \cap B^C)$.
- Use the fact that $A \perp\!\!\!\perp B$, which implies $P(A \cap B) = P(A)\,P(B)$. Substitute in the expression for $P(A \cap B^C)$, and take out the common factor P(A).

b. if A and B are independent, then so are A^C and B^C.

Hint: Use the result from part **a**.

Follow the steps below:

Strategy:

- In order to show $A^C \perp\!\!\!\perp B^C$, it is enough to show that $P(A^C \cap B^C) = P(A^C)\,P(B^C)$ as this will imply that, by definition, $A^C \perp\!\!\!\perp B^C$.
- $P(A^C \cap B^C)=?$
- We can express this from the total probability definition for B^C (of for A^C):

$P(B^C) = P(B^C \cap A) + P(B^C \cap A^C)$ (as A and A^C are two disjoint sets with probabilities summing up to 1).

$\Leftrightarrow P(B^C \cap A^C) = P(B^C) - P(B^C \cap A)$

- Use the fact $A \perp\!\!\!\perp B^C$ (implied from Problem 6a), which implies $P(A \cap B^C) = P(A)\,P(B^C)$. Substitute in the expression for $P(B^C \cap A^C)$, and take out the common factor $P(B^C)$.

Problem 8

Show that for any two events A and B with P(A)>0 and P(B)>0 the following holds: if A and B are independent, they cannot be mutually exclusive. (This is the opposite of what you proved in sections).

Hint: suppose that A and B are mutually exclusive, and show that this leads to a contradiction.

Problem 9

Chevalier de Mere's Problem

(based on "Probability Demystified", by Allan Bluman, Ch. 1 "Brief History of Probability")

During the mid-1600s, a professional gambler named Chevalier de Mere made a considerable amount of money on a gambling game. He would bet that in four rolls of a die, he could obtain at least one 6. Chevalier de Mere was winning the game more than half of the time, so people suspected him of cheating and refused to play. He then decided to invent a new game – he would bet that if he rolled a pair of dice 24 times, he would get at least one double 6. However, to his dismay, he started losing more often than he won, and lost money. Unable to figure out why he was losing, he asked the renowned French mathematician Blaise Pascal (1623–1662) to study the game. Because of this problem, Pascal became interested in studying probability and began a correspondence with his French fellow-mathematician Pierre de Fermat (1601–1665). Together the two of them were able to solve Chevalier de Mere's dilemma and formulated the foundations of probability theory.

a. Consider the first game Chevalier de Mere invented. Find the probability of obtaining at least one 6 in four rolls of a fair 6-sided die.

Hint: Denote by A_i the event *"a 6 occurs of the i-th die roll".* Then
P(*at least one 6 occurs in 4 rolls*) = P($A_1 \cup A_2 \cup A_3 \cup A_4$) = 1 – P(*no 6 occurs in 4 rolls*)=
= 1 - P($A_1^C \cap A_2^C \cap A_3^C \cap A_4^C$) = 1- P($A_i^C$)^4 because of independence.

b. Now consider Chevalier de Mere's second game. Find the probability of obtaining at least one pair of 6s when rolling a pair of fair 6-sided dice 24 times.

Hint: Denote by A_i the event *"a 6 occurs on die i"*, where $i=1,2$. Then
P(*at least one pair of 6s occurs in 24 rolls*) = 1 – P(*no pair of 6s occurs in 24 rolls*)=
= 1 – P(*no pair of 6s occurs in roll 1* ∩ *no pair of 6s occurs in roll 2* ∩ … ∩ *no pair of 6s occurs in roll 24)*
= 1 – {P(*no pair of 6s occurs in roll j*)}^{24} as each roll of a pair of dice is independent of the rest.
And, of course, the probability that we don't get a pair of 6s when a rolling two dice is $\frac{35}{36}$. (Why?)