Any homogenous heat equation with non-homogenous Dirichlet\Neumann boundary conditions can be easily converted to a non-homogenous heat equation with homogeneous Dirichlet\Neumann boundary conditions.

So we shall look at the general non-homogenous heat equation with Dirichlet BC.

In this case, the system looks like:

$$\begin{cases} u_t = \kappa^2 u_{xx} + F(x,t) \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases} \tag{1.1}$$

The associated homogenous system is

$$\begin{cases} \tilde{u}_t = \kappa^2 \tilde{u}_{xx} \\ \tilde{u}(0,t) = \tilde{u}(L,t) = 0 \\ \tilde{u}(x,0) = f(x) \end{cases} \tag{1.2}$$

We know, using separation of variables, that the homogenous system gives rise to a set of orthogonal eigenfunctions

$$\xi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \tag{1.3}$$

For brevity and generality sake we will just note that the eigenfunctions satisfy

$$\int_0^L \xi_n(x)\xi_m(x)\,dx = \begin{cases} 0 & n \neq m \\ \alpha_n & n = m \end{cases} \tag{1.4}$$

Where α is a constant.

Furthermore

$$\xi_n'' = -\lambda_n^2 \xi_n \tag{1.5}$$

For example, in the context of homogenous Dirichlet BC we see that $\alpha \doteq L/2$ and $\lambda_n \doteq n\pi/L$

And usually at this point we would find the expansion of the initial condition in terms of the eigenfunctions and equate coefficients.

However, in this case we have to look at the non-homogenous system.

We now assume that the non-homogenous solution is in the form:

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\xi_n(x) \tag{1.6}$$

Note that now the coefficient $c_n(t)$ is yet-to-be defined function of *t*.

When substituted back into the non-homogenous system we obtain:

$$u_t = \kappa^2 u_{xx} + F(x,t)$$

$$\sum_{n=1}^{\infty} c_n'(t)\xi_n(x) = \kappa^2 \sum_{n=1}^{\infty} c_n(t)\xi_n''(x) + F(x,t)$$

$$\sum_{n=1}^{\infty}\left[c_n'(t)\xi_n(x) - \kappa^2 c_n(t)\xi_n''(x)\right] = F(x,t) \tag{1.7}$$

Since $\xi_n'' = -\lambda_n^2\xi_n$ equation (1.7) can be written as:

$$\sum_{n=1}^{\infty}\left[c_n'(t) + \kappa^2\lambda_n^2 c_n(t)\right]\xi_n(x) = F(x,t) \tag{1.8}$$

Now, if we expand $F(x,t)$ in terms of the eigenfunctions:

$$F(x,t) = \sum_{n=1}^{\infty} a_n(t)\xi_n(x) \tag{1.9}$$

Where

$$a_n(t) = \frac{1}{\alpha_n} \int_0^L F(x,t)\, \xi_n(x)\, dx \tag{1.10}$$

Equation (1.8) becomes:

$$\sum_{n=1}^{\infty} \left[c_n'(t) + \kappa^2 \lambda_n^2 c_n(t) \right] \xi_n(x) = F(x,t) = \sum_{n=1}^{\infty} a_n(t)\, \xi_n(x) \tag{1.11}$$

So if we equate coefficients on both sides we get a set of n first order ordinary differential equations for $c(t)$:

$$\frac{dc_n(t)}{dt} + \kappa^2 \lambda_n^2 c_n(t) = a_n(t) \tag{1.12}$$

Since this is a first order equation we need an initial condition. This can only comes from the initial condition of the original heat equation system:

$$u(x,0) = f(x) \tag{1.13}$$

This gives:

$$u(x,0) = \sum_{n=1}^{\infty} c_n(0)\, \xi_n(x) = f(x) \tag{1.14}$$

So if we expand the initial condition in terms of the eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} b_n \xi_n(x) \tag{1.15}$$

Where the constants b_n are:

$$b_n = \frac{1}{\alpha_n} \int_0^{\infty} f(x)\, \xi_n(x)\, dx \tag{1.16}$$

Equating the coefficients of (1.14) to that of (1.16) gives

$$c_n(0) = b_n \tag{1.17}$$

And these are the initial conditions to the set of the ordinary differential equations (1.12) that can be solved using integration factor:

$$c_n' + \kappa^2\lambda_n^2 c_n' = a_n(t)$$

$$c_n' e^{\kappa^2\lambda_n^2 t} + \kappa^2\lambda_n^2 c_n' e^{\kappa^2\lambda_n^2 t} = a_n(t) e^{\kappa^2\lambda_n^2 t}$$

$$\frac{d}{dt}\left[c e^{\kappa^2\lambda_n^2 t}\right] = a_n(t) e^{\kappa^2\lambda_n^2 t}$$

$$c e^{\kappa^2\lambda_n^2 t} = \int_0^t a_n(\tau) e^{\kappa^2\lambda_n^2 \tau} d\tau$$

$$c(t) = e^{-\kappa^2\lambda_n^2 t} \int_0^t a_n(\tau) e^{\kappa^2\lambda_n^2 \tau} d\tau + A e^{-\kappa^2\lambda_n^2 t} \tag{1.18}$$

Where A is an integration constant.

And when we apply the initial conditions (1.17) we get

$$\boxed{c(t) = e^{-\kappa^2\lambda_n^2 t} \int_0^t a_n(\tau) e^{\kappa^2\lambda_n^2 \tau} d\tau + b_n e^{-\kappa^2\lambda_n^2 t}} \tag{1.19}$$

So the general solution of the non-homogenous equation with homogenous boundary conditions

$$\begin{cases} u_t = \kappa^2 u_{xx} + F\left(x,t\right) \\ u\left(x,0\right) = f\left(x\right) \end{cases} \tag{1.20}$$

is:

$$u\left(x,t\right) = \sum_{n=1}^{\infty} c_n\left(t\right)\xi_n\left(x\right)$$

$$\boxed{u\left(x,t\right) = \sum_{n=1}^{\infty}\left[e^{-\kappa^2\lambda_n^2 t}\int_0^t a_n\left(\tau\right)e^{\kappa^2\lambda_n^2\tau}d\tau + b_n e^{-\kappa^2\lambda_n^2 t}\right]\xi_n\left(x\right)} \tag{1.21}$$

Where

$$\alpha_n \doteq \int_0^L \left[\xi_n\left(x\right)\right]^2 dx \tag{1.22}$$

And:

$$a_n\left(t\right) = \frac{1}{\alpha_n}\int_0^L F\left(x,t\right)\xi_n\left(x\right)dx \tag{1.23}$$

And

$$b_n = \frac{1}{\alpha_n}\int_0^{\infty} f\left(x\right)\xi_n\left(x\right)dx \tag{1.24}$$

Recap

The system is $u_t = \kappa^2 u_{xx} + F(x,t)$ with *homogenous* (could be mixed) boundary conditions and initial condition $u(x,0) = f(x)$

- Find the eigenfunctions of the associate homogenous system $\{\xi_n(x)\}$ that satisfy the boundary conditions. Define $\alpha_n \doteq \int_0^L [\xi_n(x)]^2 \, dx$

- Expand the non-homogenous forcing term $F(x,t)$ into a series of the eigenfunctions $F(x,t) = \sum_{n=1}^{\infty} a_n(t)\xi_n(x)$ where $a_n(t) = \frac{1}{\alpha}\int_0^L F(x,t)\xi_n(x)dx$

- Expand the initial condition $u(x,0) = f(x)$ in terms of the eigenfunctions $f(x) = \sum_{n=1}^{\infty} b_n(t)\xi_n(x)$ where $b_n(t) = \frac{1}{\alpha}\int_0^L f(x)\xi_n(x)dx$

- Solve the set of differential equations $c_n' + \kappa^2\lambda_n^2 c_n = a_n$ with the initial condition $c_n(0) = b_n$ The solution of such a differential equation is $c(t) = e^{-\kappa^2\lambda_n^2 t}\int_0^t a_n(\tau)e^{\kappa^2\lambda_n^2\tau}d\tau + b_n e^{-\kappa^2\lambda_n^2 t}$

- The solution is $u(x,t) = \sum_{n=0}^{\infty} c_n(t)\xi_n(x)$

Special case: Time independent forcing term

If $F(x,t) = F(x)$, that is the forcing term is time independent, then

$a_n = \frac{1}{\alpha}\int_0^L F(x)\xi_n(x)dx$ is a constant.

This gives:

$$c(t) = a_n e^{-\kappa^2\lambda_n^2 t}\int_0^t e^{\kappa^2\lambda_n^2\tau}d\tau + b_n e^{-\kappa^2\lambda_n^2 t}$$

$$c(t) = \frac{a_n}{\kappa^2\lambda_n^2} + \left(b_n - \frac{a_n}{\kappa^2\lambda_n^2}\right)e^{-\kappa^2\lambda_n^2 t} \tag{1.25}$$

And

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[\frac{a_n}{\kappa^2\lambda_n^2} + \left(b_n - \frac{a_n}{\kappa^2\lambda_n^2}\right)e^{-\kappa^2\lambda_n^2 t}\right]\xi_n(x)} \tag{1.26}$$

We see that indeed the initial condition is satisfied:

$$u(x,0) = \sum_{n=1}^{\infty} b_n\xi_n(x) = f(x) \tag{1.27}$$

And as $t \to \infty$ we get the steady state:

$$u_E = \lim_{t\to\infty} u(x,t) = \sum_{n=1}^{\infty}\frac{a_n}{\kappa^2\lambda_n^2}\xi_n(x) \tag{1.28}$$

Recall that the steady state must satisfy

$$u_t = \frac{d^2 u_E}{dx^2} + F(x) = 0 \tag{1.29}$$

And from (1.28) we get

$$\kappa^2 \frac{d^2 u_E}{dx^2} + F(x) = 0$$

$$\kappa^2 \sum_{n=1}^{\infty} \frac{a_n}{\kappa^2 \lambda_n^2} \xi_n''(x) + F(x) = 0 \qquad\qquad \xi_n'' = -\lambda_n^2 \xi_n$$

$$-\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^2} \lambda_n^2 \xi_n + F(x) = 0$$

$$F(x) = \sum_{n=1}^{\infty} a_n \xi_n \tag{1.30}$$

Which is the definition of the expansion of the forcing term. Also since $\xi_n(x)$ satisfies the boundary conditions, so is the steady state.

Converting a non-homogenous equation with time-independent forcing term to a homogenous equation.

The most general case with homogenous boundary conditions is:

$$\begin{aligned}
&u_t = \kappa^2 u_{xx} + F(x) \\
&\alpha_1 u(0,t) + \beta_1 u_x(0,t) = \gamma_1 \\
&\alpha_2 u(L,t) + \beta_2 u_x(L,t) = \gamma_2 \\
&u(x,0) = f(x)
\end{aligned} \tag{1.31}$$

where α_i, β_i are constants, and γ_1, γ_2 may be time dependent functions.

We solve for the steady state solution

$$\kappa^2 \frac{d^2 u_E}{dx^2} + F(x) = 0 \tag{1.32}$$

That satisfies the above boundary conditions:

$$\begin{aligned}
&\alpha_1 u_E(0) + \beta_1 u_E'(0) = \gamma_1 \\
&\alpha_2 u_E(L) + \beta_2 u_E'(L) = \gamma_2
\end{aligned} \tag{1.33}$$

Then we define

$$u(x,t) = v(x,t) + u_E \tag{1.34}$$

When we apply it to the equation we get:

$$\begin{aligned}
&u_t = v_t \\
&u_{xx} = v_{xx} + \frac{d^2 u_E}{dx^2}
\end{aligned} \tag{1.35}$$

But from (1.32) we get

$$u_{xx} = v_{xx} + \frac{d^2 u_E}{dx^2} = v_{xx} - \frac{F(x)}{\kappa^2} \tag{1.36}$$

Thus:

$$u_t = \kappa^2 u_{xx} + F(x)$$

$$v_t = \kappa^2 \left[v_{xx} - \frac{F(x)}{\kappa^2} \right] + F(x)$$

$$v_t = \kappa^2 v_{xx} - F(x) + F(x)$$

$$v_t = \kappa^2 v_{xx} \tag{1.37}$$

The first boundary condition becomes

$$\alpha_1 u(0,t) + \beta_1 u_x(0,t) = \gamma_1$$

$$\alpha_1 v(0,t) + \alpha_1 u_E(0) + \beta_1 v_x(0,t) + \beta_1 u_E'(0) = \gamma_1$$

$$\alpha_1 v(0,t) + \beta_1 v_x(0,t) + \left[\alpha_1 u_E(0) + \beta_1 u_E'(0) \right] = \gamma_1 \tag{1.38}$$

But since u_E satisfies the original boundary conditions then $\alpha_1 u_E(0) + \beta_1 u_E'(0) = \gamma_1$ we get:

$$\alpha_1 v(0,t) + \beta_1 v_x(0,t) = 0 \tag{1.39}$$

And by the same token:

$$\alpha_2 v(L,t) + \beta_2 v_x(L,t) = 0 \tag{1.40}$$

The initial condition is now:

$$u(x,0) = f(x)$$

$$v(x,0) + u_E(x) = f(x)$$

$$v(x,0) = f(x) - u_E(x) \doteq g(x) \tag{1.41}$$

So the system now becomes homogenous:

$$\begin{aligned} & v_t = \kappa^2 v_{xx} \\ & \alpha_1 v(0,t) + \beta_1 v_x(0,t) = 0 \\ & \alpha_2 v(L,t) + \beta_2 v_x(L,t) = 0 \\ & v(x,0) = g(x) \end{aligned} \tag{1.42}$$

To Recap:

To convert the system

$$\begin{aligned} & u_t = \kappa^2 u_{xx} + F(x) \\ & \alpha_1 u(0,t) + \beta_1 u_x(0,t) = \gamma_1 \\ & \alpha_2 u(L,t) + \beta_2 u_x(L,t) = \gamma_2 \\ & u(x,0) = f(x) \end{aligned}$$

To a homogeneous system:

- Find the steady state $u_E(x)$ that solves $\kappa^2 u_E'' + F(x) = 0$ and satisfies the boundary conditions $\alpha_1 u_E(0) + \beta_1 u_E'(0) = \gamma_1$ and $\alpha_2 u_E(L) + \beta_2 u_E'(L) = \gamma_2$
- Define $v(x,t) = u(x,t) - u_E$
- Define $v(x,0) = f(x) - u_E = g(x)$

The system becomes

$$\begin{aligned} & v_t = \kappa^2 v_{xx} \\ & \alpha_1 v(0,t) + \beta_1 v_x(0,t) = 0 \\ & \alpha_2 v(L,t) + \beta_2 v_x(L,t) = 0 \\ & v(x,0) = g(x) \end{aligned}$$

The solution to the original system is $u(x,t) = v(x,t) + u_E$

Example:

the system

$$\begin{aligned}
&u_t = u_{xx} + x \\
&u(0,t) + u_x(0,t) = 0 \\
&u(1,t) - u_x(1,t) = 1 \\
&u(x,0) = 2x^2
\end{aligned} \tag{1.43}$$

First, we need to solve for the steady state:

$$u_t = u_E'' + x = 0 \quad \Rightarrow \quad u_E'' = -x \tag{1.44}$$

Integrating once:

$$u_E' = -\frac{x^2}{2} + A$$

Integrating twice:

$$u_E = -\frac{x^3}{6} + Ax + B \tag{1.45}$$

Applying boundary conditions:

$$\begin{aligned}
u_E &= -\frac{x^3}{6} + Ax + B \\
u_E' &= -\frac{x^2}{2} + A
\end{aligned}$$

$$\begin{cases} u_E(0) + u_E'(0) = B + A = 0 \\ u_E(1) - u_E'(0) = -\frac{1}{6} + A + B + \frac{1}{2} - A = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} A = -B \\ B + \frac{1}{3} = 1 \end{cases}$$

$$B = -A = \frac{2}{3}$$

$$u_E = -\frac{x^3}{6} + Ax + B = -\frac{x^3}{6} - \frac{2x}{3} + \frac{2}{3} \tag{1.46}$$

Or:

$$u_E = \frac{4 - 4x - x^3}{6} \tag{1.47}$$

So if we set

$$u(x,t) = v(x,t) + u_E = v(x,t) + \frac{4 - 4x - x^3}{6} \tag{1.48}$$

We get a homogenous equation with respect to $v(x,t)$

$$\begin{cases} u_t = v_t \\ u_{xx} = v_{xx} - x \end{cases} \quad \Rightarrow \quad u_t = u_{xx} + x \quad \Rightarrow \quad v_t = v_{xx} \tag{1.49}$$

The boundary conditions for $v(x,t)$ are

$$u(0,t) + u_x(0,t) = 0$$

$$v(0,t) + u_E(0) + v_x(0,t) + u'_E(0) = 0$$

$$v(0,t) + v_x(0,t) + \underbrace{\left[u_E(0) + u'_E(0)\right]}_{=0} = 0$$

$$v(0,t) + v_x(0,t) + \underbrace{\left[u_E(0) + u'_E(0)\right]}_{=0} = 0 \tag{1.50}$$

And:

$$u(1,t) - u_x(1,t) = 1$$

$$v(1,t) + u_E(1) - v_x(1,t) - u'_E(1) = 1$$

$$v(1,t) - v_x(1,t) + \underbrace{\left[u_E(1) - u'_E(1)\right]}_{=1} = 1$$

$$v(1,t) - v_x(1,t) = 0 \tag{1.51}$$

So far we have

$$\begin{cases} v_t = v_{xx} \\ v(0,t) + v_x(0,t) = 0 \\ v(1,t) - v_x(1,t) = 0 \end{cases} \tag{1.52}$$

All that is left is to transform the initial condition:

$$u(x,0) = f(x) = 2x^2$$

$$u(x,0) = v(x,0) + u_E = f(x)$$

$$v(x,0) = g(x) = f(x) - u_E$$

$$v(x,0) = g(x) = 2x^2 - \frac{4 - 4x - x^3}{6}$$

$$g(x) = \frac{x^3 + 12x^2 + 4x - 4}{6} \tag{1.53}$$

The system is now:

$$\begin{cases} v_t = v_{xx} \\ v(0,t) + v_x(0,t) = 0 \\ v(L,t) - v_x(L,t) = 0 \\ v(x,0) = \dfrac{x^3 + 12x^2 + 4x - 4}{6} \end{cases} \tag{1.54}$$

Which is completely homogeneous.

And the solution to the original system is

$$u(x,t) = v(x,t) + u_E = v(x,t) + \frac{4 - 4x - x^3}{6} \tag{1.55}$$