8. For a particle of mass m moving in a one-dimensional potential of the form $V(x) = \frac{1}{2}m\omega^2x^2$, the operator $\hat{A}$ and its hermitian conjugate, $\hat{A}^\dagger$, are defined to be

$$\hat{A} = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} + \mathrm{i}\hat{p}\right) \quad \text{and} \quad \hat{A}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} - \mathrm{i}\hat{p}\right).$$

Show that the Hamiltonian of the system may be written

$$\hat{H} = \left(\hat{A}^\dagger\hat{A} + \tfrac{1}{2}\right)\hbar\omega.$$

Prove the following commutation relations:

$$[\hat{A}, \hat{A}^\dagger] = 1 \quad \text{and} \quad [\hat{H}, \hat{A}] = -\hbar\omega\hat{A}.$$

Using these results, or otherwise, show that if $|n\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue E_n, $\hat{A}\,|n\rangle$ is also an eigenstate of $\hat{H}$, and find its energy eigenvalue. Find the energy of the lowest-lying level, and hence show that $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, where $n = 0, 1, 2, \ldots$.

By writing the operators $\hat{x}$ and $\hat{p}$ in terms of $\hat{A}$ and $\hat{A}^\dagger$, or otherwise, show that if the oscillator is in the eigenstate $|n\rangle$ the product of the uncertainty in momentum and position is given by

$$\Delta x \Delta p = (n + \tfrac{1}{2})\hbar,$$

where for any operator $\hat{Q}$,

$$(\Delta Q)^2 = \langle n|\,\hat{Q}^2\,|n\rangle - \left(\langle n|\,\hat{Q}\,|n\rangle\right)^2.$$

Comment on this result.

[You may assume that $\hat{A}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$.]