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The Meaning of Internal Rates of Return

ROBERT DORFMAN

ABSTRACT

Nearly one hundred years after Irving Fisher's persuasive argument that net present value is the fundamental criterion for appraising investment projects, businessmen and bankers continue to consider the internal rate of return. Business practice is justified in some circumstances. It has long been recognized that a firm will grow asymptotically at a rate equal to the largest real positive root of an individual project's rate of return equation if the net cash flows are continually reinvested in projects of the same type. That same root also controls the firm's asymptotic growth rate if any fixed proportion of the cash flows is reinvested. The other roots of the equation are important also, since the stability of the firm's growth path depends on them.

FOR A VERY LONG time, two families of criteria for capital investment decisions have coexisted: net present value criteria, most forcefully advanced by Irving Fisher [9], and internal rate of return criteria. Throughout most of this time the Fisherian criteria have received the endorsement of orthodox economic theory, but the rate of return criteria have survived in business and banking practice, as attested by the fact that the most popular financial hand calculator contains a built-in program for computing rates of return. The purpose of this paper is to explore the element of validity in the rate of return criteria and, secondarily, to clarify the meaning of the internal rates of return—there are almost always several.

The debate between net present value criteria and internal rate of return criteria goes back to the inception of modern interest theory. A definitive statement of the case for net present value criteria appears in Irving Fisher's *The Rate of Interest* [8]; that argument remains the foundation of the dominant school of capital investment appraisal to this day. The internal rate of return criterion is implicit in Boehm-Bawerk's *Positive Theorie des Kapitales* [3]. There, Boehm-Bawerk took it for granted that businessmen would (and should) invest so as to obtain the greatest annual net cash flow in perpetuity per dollar invested—a simple version of the internal rate of return principle. This approach has remained popular ever since (though not so popular among theorists, with some exceptions including Keynes [15]). The list of intervening contributors, expositors, and commentators is too long to be worth recounting. A few have to be mentioned, however.

Alchian [1] clarified the conceptual relationships among the internal rate of return, the net present value criterion, and a subsidiary concept, Fisher's "rate of return over cost". Hirschleifer's influential paper [13] explored the implications of imperfections in the capital markets for the net present value criterion. Marglin, in a pair of elegant articles [16, 17], showed how to incorporate the possibility of reinvesting proceeds in the present value criterion. Wright [19], Fleming and

Wright [10], and, independently Arrow and Levhari [2] threw light on the most confusing technical aspect of the internal rate of return criterion by showing that if the duration of an investment project is subject to choice and if it is chosen to maximize the internal rate of return, then the equation for the rate of return can have only one real root.

The strongest influence on the present paper is the work of John S. Chipman. In a series of papers [5, 6] he developed the insights that when the net proceeds resulting from investments in an economy are wholly or partially reinvested, the growth of the economy can be described by a renewal equation, and, under some appropriate assumptions, the formula for the roots of that equation is identical with the rate of return equation for the typical investment in that economy.

The net present value criterion is supported by a powerful argument. The argument has its limitations, however, three of which are relevant in the present context. First, in its original and fundamental form, the present value is computed from the cash flows generated by an initial act of investment, without allowance for the results of possible reinvestment of those cash flows or even full maintenance and replacement of the physical capital originally procured. Beside ignoring some important consequences of an investment, this convention makes it difficult to compare undertakings with different economic lives. To take the starkest example, two alternative investment opportunities cost \$1,000 each; the first yields \$1,100 in one year, the second yields \$1,166 in two years, the market rate of interest is 5%. Straightforwardly, the net present value of the first is \$48, that of the second is \$58; the second is preferable. But which leaves the investor better off at the end of two years? On the Fisherian assumption, he will have to reinvest his \$1,100 at the market rate and end with \$1,155, which is certainly inferior to the second alternative's result. Alternatively, the investor might be able to repeat his investment after the first year, and obtain \$1,210 at the end of the second. Which assumption to make is obscure, and with that obscurity the decisiveness of the net present value argument fades. (See Hildreth [12]).

Later work (Galenson and Leibenstein [11], Eckstein [7], Marglin [16] and [17]) has corrected this deficiency. One has only to impose a reinvestment and replacement policy and then to include in the calculation the cash flows attributable to the daughter investments, and their daughters, and so on forever. Of course, the assumptions about the original investment's progeny have to be pretty tenuous. But, as the example shows, to ignore them is to assume that after the initial investment no opportunities more promising than the market rate of interest will become available, which is not a very appealing assumption.

The second limitation is that the Fisherian argument presumes perfect financial markets. The "separation theorem" rests on this assumption. Because of it, the precise pattern of cash flows is irrelevant; any pattern can be exchanged in the financial markets for any other of the same present value. Hirschleifer [13] has made clear the consequences of dropping it; the pattern of cash flows can then matter a great deal, and the net present value is an incomplete criterion for investment appraisal.

Those two limitations weaken the case for the net present value criterion; the third limitation cuts deepest. Fisher's whole structure is an elaboration of the tension between opportunities to invest and impatience to consume. It assumes that the purpose of investing is to be able to afford the greatest possible amount of consumption. (In the presence of the perfect financial market assumption, the greatest amount of consumption is well-defined as the consumption pattern with greatest possible present value). If the purpose of investment is anything other than enhancing the ability to consume, then some other criterion may well be appropriate. The literatures of both corporate behavior and economic development suggest that quite frequently promoting consumption is not the dominant goal. A number of plausible goals have been hypothesized and supported with more or less evidence, so that one tends to conclude that the purpose to be served by investment is not the same in all instances, and that the appropriate investment criterion varies correspondingly. One particularly appealing alternative objective is growth: growth of the enterprise or growth of the economy as the case may be. In the sequel, we shall develop the implications of the maximum growth objective, and shall see that in conjunction with some assumptions about reinvestment opportunities, it entails an internal rate of return criterion for selecting investments. We shall first develop this thesis in the context of a firm that desires to grow as rapidly as possible by using internally generated funds.

I. Investment Appraisal for Internally Financed Projects

We now develop a rule for project selection for a firm that desires to grow as rapidly as possible. In this section we assume that this growth is to be financed entirely by retained earnings and depreciation charges. But not all retained earnings are available for reinvestment. A certain proportion of them is claimed by income taxes and another proportion has to be paid out as dividends. The residue, say proportion θ of retained earnings after depreciation, will be reinvested along with the entire amount charged to depreciation. These assumptions enable us to formualte the growth path that will result from a policy of investing in undertakings of any specified type.

A particular type of investment project can be characterized adequately for our purposes by specifying the net cash flow that it generates, per dollar of initial investment, in each year of its economic life. The net cash flow in the τth year will be denoted by $f(\tau)$. For example, if a type of investment costs 1,000 initially and generates successively cash flows of 3,250, − 6,500, 5,000 in its three-year life it would be characterized by the table

τ	$f(\tau)$
1	3.25
2	−6.50
3	5.00

In each year also, the investment depreciates. Depreciation in year τ will be denoted by $\delta(\tau)$. Then, following the rule of reinvesting each year the depreciation charge plus the proportion of θ of cash flow after depreciation, the reinvestment generated in the τth year of the project's life per dollar of initial investment will be

$$\begin{aligned} y(\tau) &= \delta(\tau) + \theta(f(\tau) - \delta(\tau)) \\ &= \theta f(\tau) + (1 - \theta)\delta(\tau) \qquad (1) \end{aligned}$$

We assume that the daughter projects have the same cash flows as the parent, and denote by $Y(t)$ the total amount of gross investment in calendar year t. This will be the total amount of investment that will result from projects initiated 1, 2, . . . , T years previously, where T is the length of the individual project's life. Symbolically,

$$Y(t) = \sum_{\tau=1}^{T} Y(t-\tau)y(\tau) \tag{2}$$

The $y(\tau)$ are simply constants, determined by Equation (1). Then Equation (2) is a Tth order difference equation in $Y(t)$, with constant coefficients. The solution of such an equation has the form

$$Y(t) = \sum_{i=1}^{T} c_i(1+u_i)^t \tag{3}$$

on the assumption (which avoids a good deal of algebraic complexity) that all the roots $(1 + u_i)$ are distinct. The roots $(1 + u_i)$ are the solution to the polynomial equation

$$\sum_{\tau=1}^{T} \frac{y(\tau)}{(1+u_i)^{\tau}} = 1 \tag{4}$$

Notice the similarity of this equation to the internal rate of return equation for the individual projects. The only difference is that $y(\tau)$ appears in the numerator instead of $f(\tau)$. The c_i are constants which are determined by any T values of $Y(t)$ along the growth path, usually the first T.

Our remaining task is to determine the roots $(1 + u_i)$, which depend on the relations among $y(\tau)$, $f(\tau)$, and $\delta(\tau)$. To this end, the depreciation formula must be specified. It appears that only one depreciation formula leads to a tractable solution. This is the formula for "economic depreciation" introduced by Hotelling [14].

It goes as follows. Let $V(\tau)$ denote the value of the assets per dollar of initial investment at the end of the τth year of the project's life. By convention, the initial value is denoted $V(0)$ and, by definition, $V(0) = 1$. The value $V(\tau)$, at the end of any year is the present value of future cash flows discounted at an appropriate rate of interest, r, or

$$V(\tau) = \sum_{s=\tau+1}^{T} \frac{f(s)}{(1+r)^{s-\tau}} \tag{5}$$

At once $V(T) = 0$. Applying this formula to $\tau = 0$:

$$V(0) = \sum_{1}^{T} \frac{f(s)}{(1+r)^{s}} = 1 \tag{6}$$

Thus, r can be any root of the internal rate of return equation. We leave in abeyance for the moment which root should be chosen.

The depreciation formula follows at once. Depreciation in the τth year of a project is the decrease in the value of the assets during that year, or

$$\delta(\tau) = V(\tau-1) - V(\tau)$$

By comparing Equation (5) for $V(\tau)$ and $V(\tau - 1)$, we obtain the recursion relation

$$V(\tau) = (1+r)V(\tau-1) - f(\tau)$$

and thence

$$\delta(\tau) = f(\tau) - rV(\tau - 1) \tag{7}$$

The economic commonsense of these formulas is apparent.

Inserting the formula for $\delta(\tau)$ and Equation (5) in Equation (1) we obtain

$$\begin{aligned} y(\tau) &= f(\tau) - (1 - \theta)rV(\tau - 1) \\ &= f(\tau) - (1 - \theta)r \sum_{s=\tau}^{T} \frac{f(s)}{(1 + r)^{s-\tau+1}} \end{aligned}$$

We are now in a position to determine the roots of Equation (4). Insert the equation for $y(\tau)$ in that equation to obtain

$$\sum_{\tau=1}^{T} \frac{f(\tau)}{(1 + u_i)^{\tau}} - (1 - \theta) \frac{r}{1 + r} \sum_{\tau=1}^{T} \frac{1}{(1 + u_i)^{\tau}} \sum_{s=\tau}^{T} \frac{f(s)}{(1 + r)^{s-\tau}} = 1 \tag{8}$$

The double sum is

$$\begin{aligned} \sum_{\tau=1}^{T} \left(\frac{1 + r}{1 + u_i}\right)^{\tau} \sum_{s=\tau}^{T} \frac{f(s)}{(1 + r)^{s}} &= \sum_{s=1}^{T} \frac{f(s)}{(1 + r)^{s}} \sum_{\tau=1}^{s} \left(\frac{1 + r}{1 + u_i}\right)^{\tau} \\ &= \sum_{s=1}^{T} \frac{f(s)}{(1 + r)^{s}} (1 + r) \frac{1 - \left(\frac{1 + r}{1 + u_i}\right)^{s}}{u_i - r} \\ &= (1 + r) \sum_{s=1}^{T} f(s) \frac{(1 + r)^{-s} - (1 + u_i)^{-s}}{u_i - r}, \quad u_i \neq r \end{aligned}$$

Using this fact, the equation becomes

$$\sum_{1}^{T} \frac{f(\tau)}{(1 + u_i)^{\tau}} - \frac{(1 - \theta)r}{u_i - r} \sum_{1}^{T} f(\tau)[(1 + r)^{-\tau} - (1 + u_i)^{-\tau}] = 1, \; u_i \neq r$$

or

$$\frac{u_i - \theta r}{u_i - r} \sum_{1}^{T} \frac{f(\tau)}{(1 + u_i)^{\tau}} - \frac{(1 - \theta)r}{u_i - r} \sum_{1}^{T} \frac{f(\tau)}{(1 + r)^{\tau}} = 1$$

We can solve this equation by inspection. Denote the roots of Equation (6) by $r_1, r_2, \cdots, r_T$ with r_1 = the smallest real positive root. Choose $r = r_1$. Then the second summation above becomes unity and the whole equation becomes

$$\frac{u_i - \theta r_1}{u_i - r_1} \sum_{1}^{T} \frac{f(\tau)}{(1 + u_i)^{\tau}} = \frac{u_i - \theta r_1}{u_i - r_1}$$

This equation is obviously satisfied by $u_1 = \theta r_1$ and also by $u_i = r_i$, $i = 2, 3, \cdots, T$, for then the summation equals unity.

We now have the solution to Equation (3). It is

$$Y(t) = c_1(1 + \theta r_1)^t + \sum_{i=2}^{T} c_i(1 + r_i)^t \tag{9}$$

In words: the growth path of investment, and thereby of total cash flows, dividends, and capital assets, is a sum of exponentials, one of which grows at the rate θr_1, and the others at the rates given by the other roots of the internal rate of return equation.

To see clearly what this formula is telling us, we perform the calculation for the three-period illustration given at the outset, which was constructed to be both transparent and ill-behaved. The internal rates of return are readily found to be

$$r_1 = \tfrac{1}{4},\ r_2 = i\sqrt{3},\ r_3 = -i\sqrt{3}\ (i = \sqrt{-1})$$

Then the growth path resulting from investing in this type of undertaking and reinvesting in the same type is

$$Y(t) = c_1(1 + \tfrac{1}{4}\,\theta)^t + c_2(1 + i\sqrt{3})^t + c_3(1 - i\sqrt{3})^t$$

It is convenient to write

$$1 + i\sqrt{3} = 2\ (\cos 60 + i \sin 60)$$

and similarly for $1 - i\sqrt{3}$. Then, for instance,

$$(1 + i\sqrt{3})^t = 2^t(\cos 60t + i \sin 60t)$$

Furthermore, for the growth path to consist of real numbers, c_2 and c_3 must be complex complements, say $a(\cos \omega + i \sin \omega)$ and $a(\cos \omega - i \sin \omega)$. Then, doing a little algebra,

$$c_2 2^t\ (\cos 60t + i \sin 60t) = a2^t\ (\cos(60t + \omega) + i \sin(60t + \omega))$$

and

$$c_3 2^t\ (\cos 60t - i \sin 60t) = a2^t\ (\cos(60t + \omega) - i \sin(60t + \omega))$$

and finally

$$Y(t) = c_1(1 + \tfrac{1}{4}\theta)^t + 2^{t+1}\, a \cos(60t + \omega)$$

The growth path consists of an exponential term with growth rate $\theta/4$ and a cosine wave of exponentially growing amplitude. The disposable constants c_2 and c_3 have been replaced by a and ω, which specify the amplitude and phase of the trigonometric term.

Although the principal root displays a substantial rate of growth, this growth path as a whole can clearly be disastrous because of the exponentially increasing oscillations. One can draw either or both inferences. Either this type of undertaking is not a healthy steady diet for a growing firm, or else the firm must choose the initiating three levels of investment, $Y(1)$, $Y(2)$, $Y(3)$, so as to start along a growth path on which $a = 0$. The initial investments $Y(1) > 0$, $Y(2) = (1 + \theta/4)\,Y(1)$, $Y(3) = (1 + \theta/4)\,Y(2)$ will serve nicely. Thereafter, with such an unstable growth path, the firm must be watchful for random disturbances, and take prompt action to return to the exponential path if any occur.

It should be emphasized that the example was chosen to be badly behaved, so as to make clear the sort of thing that can happen in principle. Notice that each \$1,000 invested obligates the firm to invest a further \$6,500 two years later. More normal cash flow patterns do not generate such instability. They may have no complex roots at all, or all the complex roots may have amplitudes smaller than $1 + r^*$, where r^* = largest real positive root. In those more normal cases, whatever the initial investments, the firm will tend to grow asymptotically at the rate r^* if there are several real positive roots, or θr^* if there is only one.

We thus have justified the following internal rate of return criterion:

For a firm to achieve the greatest possible rate of growth by investing in a succession of similar projects, with a fixed proportion of the net cash flow of each generation used to finance the net investment in the next generation, it should choose a type of project in which the greatest positive real root of the internal rate of return equation (i.e., Equation (6)) *is as great as possible.*

Furthermore, in choosing a type of investment project, all the roots of the internal rate of return equation should be taken into account because all of them influence the potential growth path. If there is any negative or complex root with an absolute value as great as $1 + r^*$, the growth path will be intolerably unstable; such investments should be avoided no matter how great the principal root. On the other hand, if $1 + r^*$ is greater than the absolute value of any of the other roots, the enterprise will approach asymptotically an exponential growth path with the greatest growth rate achievable.

Two technicalities remain to be cleared up. First, can we be sure that there will be a positive real root? It is easy to see that a sufficient condition is $\sum f(\tau)>1$. For then, if $r = 0$, the middle member of Equation (6) is greater than unity, whereas that middle member can always be made less than unity by choosing r sufficiently large. By continuity there must be an intermediate value of r that satisfies the equation.

The other dangling technicality is the case $\theta = 1$, i.e., the case where the entire net cash flow is available for reinvestment. The Xerox Corporation during its first decade was as good an approximation of this case as is likely to be found in the real world. The demonstration given above does not apply to this case (because the restriction $u_i \neq r$ is violated). Nevertheless, Equation (9), with $\theta = 1$, remains valid. Simply notice that if $\theta = 1$ in Equation (1), $y(\tau) = f(\tau)$. Thereupon, Equation (4) becomes identical to the internal rate of return equation and has the same roots.

II. Investment Appraisal Admitting Debt-finance

The analysis can now be extended to admit debt-finance. As before, a financial and investment policy has to be specified, and only a special, though economically sound, policy leads to an intelligible analysis.

All the assumptions of the previous section will be retained except those relating to internal financing of net investment. Instead we shall assume that the firm maintains a prescribed degree of leverage by borrowing a proportion b of net investment. (Then its leverage will be $b/(1 - b)$, $0 \leq b < 1$.) To maintain the prescribed leverage, we specify also that the firm retires its debt *pari passu* with the depreciation of the underlying assets. This can be achieved, for example, by issuing serial bonds to finance the net investment. We now show that this more general case is formally the same as the previous one.

With these assumptions, the net profit earned by an investment of \$1, or an equity investment of \$(1 − b), in the τth year of its life will be

$$f(\tau) - \delta(\tau) - r^0 bV(\tau - 1)$$

where r^0 is the rate of interest on the borrowed funds. Then, if the proportion $(1 - \theta)$ of net profit is paid to stockholders and for taxes, $r^0 bV(\tau - 1)$ is paid for interest, and $b\delta(\tau)$ is used to retire debt, the contribution of a unit investment to the internal funds available for reinvestment will be

$$f(\tau) - (1-\theta)(f(\tau) - \delta(\tau) - r^0 bV(\tau-1)) - r^0 bV(\tau-1) - b\delta(\tau)$$

$$= \theta(f(\tau) - \delta(\tau) - r^0 bV(\tau-1)) + (1-b)\delta(\tau)$$

in the τth year of its life. Allowing for the fact that the proportion b of new gross investment is to be financed by borrowing, these internal funds will support an investment of

$$y(\tau) = \theta'(f(\tau) - \delta(\tau) - r^0 bV(\tau-1)) + \delta(\tau)$$

where $\theta' = \theta/(1-b)$. Now substitute for $\delta(\tau)$ by Equation (7) to obtain

$$\begin{aligned} y(\tau) &= f(\tau) + \theta'(rV(\tau-1) - r^0 bV(\tau-1)) - rV(\tau-1) \\ &= f(\tau) - ((1-\theta')r + \theta' r^0 b)V(\tau-1) \end{aligned}$$

The law of growth of the sequence of investments is again given by Equation (2), and its solution has the form of Equation (3). The equation for the roots of Equation (3) then becomes

$$\sum_1^T \frac{y(\tau)}{(1+u_i)^\tau} = \sum_1^T \frac{f(\tau)}{(1+u_i)^\tau} - ((1-\theta')r + \theta' r^0 b) \sum_1^T \frac{V(\tau-1)}{(1+u_i)^\tau} = 1$$

Remembering that

$$\sum_1^T \frac{V(\tau-1)}{(1+u_i)^\tau} = \sum_1^T f(\tau)\,\frac{(1+r)^{-\tau} - (1+u_i)^{-\tau}}{u_i - r}$$

and that r is a root of the internal rate of return equation, this simplifies to

$$\sum_1^T \frac{f(\tau)}{(1+u_i)^\tau} - \frac{(1-\theta')r + \theta' br^0}{u_i - r}\left(1 - \sum_1^T \frac{f(\tau)}{(1+u_i)^\tau}\right) = 1$$

or

$$\frac{u_i - \theta'(r - br^0)}{u_i - r}\sum_1^T \frac{f(\tau)}{(1+u_i)^\tau} = \frac{u_i - \theta'(r - br^0)}{u_i - r}, \qquad u_i \neq r \tag{10}$$

Now select one of the real positive roots of the internal rate of return equation to be used in the formula for depreciation, Equation (5). Call the selected root r_1. Then Equation (10) is satisfied when $u_i = r_i$, $i = 2, \cdots, T$, and also when

$$\begin{aligned} u_1 &= \theta'(r_1 - br^0) \\ &= \theta\frac{r_1 - br^0}{1-b} \\ &= \theta(r_1 + \frac{b}{1-b}(r_1 - r^0)) \end{aligned} \tag{11}$$

Notice that the solution obtained in the preceding section is the special case of this solution in which $b = 0$.

Notice also that for any value of b, u_1 is an increasing function of r_1 if $r_1 > r^0$. It is therefore advantageous to choose the largest positive real root, r^*, for r_1 unless $r^* \leq r^0$. In the latter case, debt finance should not be used at all.

Assuming $r_1 = r^* > r^0$, when borrowing is permitted the firm can grow more rapidly than if confined to internally generated funds in direct proportion to (a) the leverage, and (b) the excess of the largest real positive root of the internal rate of return equation over the interest rate on borrowed funds.

The foregoing discussion has presumed that b is somehow prescribed to the firm. More normally, the leverage is a matter of financial policy. If b can be chosen with no restrictions other than $0 \leq b < 1$, and if r^0 is not affected by the choice of b, both of which are unlikely, then Equation (11) shows that the firm can be made to grow as rapidly as desired, simply by choosing a large enough leverage. The analysis of the choice of the leverage in more plausible circumstances is straightforward, and is left to the reader.

III. Conclusion

We have argued that an enterprise may have any of a number of objectives in mind when it selects investment projects. If its objective is to maximize the value of its distributions, then some version of the net present value criterion is likely to be appropriate. Otherwise, the investment criterion should be designed to reflect the objective that is in view.

We paid particular attention to situations in which the objective was to maximize the enterprise's rate of growth, and found that in certain circumstances this objective was best achieved by adopting an internal rate of return criterion. The reason was that when growth is the objective, the critical consideration in choosing among opportunities is the extent to which they generate funds available for reinvestment, and the best opportunity from this point of view is not necessarily the one with the highest net present value of cash flows.

The circumstances we assumed in order to prove those assertions were implausible. We envisaged a firm like McDonald's or Fotomat that used the investable funds from each generation of investments to build more McDonald's or Fotomats with cash flows precisely the same as their predecessors, *ad infinitum*.

Referring to earlier arguments along the same lines (Boulding [4, pp. 680–81, and earlier editions], Chipman [5]), Samuelson [18] derided this argument as "far-fetched", and, indeed, it is. Farfetchedness is a characteristic of all economic theories which are simple enough to yield intelligible insights, which is why none of them ought to be taken literally. The question that continually confronts the applied theorist is which simplified "stylized model" most adequately (or least inadequately) incorporates the essential features of the real-life phenomena he is trying to understand. In the choice between the internal rate of return and the net present value criteria, the question is whether the firm or firms under consideration are more like firms which are interested primarily in growing and are in a position, for some considerable period of time, to invest in a succession of similar investment projects with similar time-paths of returns per dollar, or whether they are more like firms interested primarily in net payouts to their proprietors who have access to perfect capital markets. Both models are far-fetched enough to generate legitimate qualms, and each catches the essence of some situations. Businessmen do pay attention to the internal rate of return of prospective investment projects, and often justify doing so by emphasizing the importance of reinvestment and affirming their confidence that opportunities similar to those now available will continue to open up in the future.

Two implications of our formulation and analysis ought to be made explicit. The first is constructive. If a type of investment project has several real positive internal rates of return, the one that is relevant for decision is the greatest of them, for that rate of return corresponds to the growth rate that will dominate eventually and may dominate all along the line.

The second implication is limiting. Suppose that an enterprise has several types of investment opportunity with different rates of return and, for some reason, follows the policy of reinvesting a fixed (non-zero) proportion of its investable cash flow in each of them. Then it is easy to show that the enterprise's dominant rate of growth will not be any simple average of the dominant growth rates of the individual types of investment but, rather, will depend on all the rates of return (positive, negative, and complex) of all the types of investment. It follows that the analysis cannot be extended to enterprises which diversify their undertakings. In essence, such an enterprise sacrifices its opportunity to grow at a maximal rate in the interest of diversification, and the extent of the sacrifice depends in a very complicated way on all the terms of the growth paths of the individual types of investment.

Many significant questions remain unanswered. The most pressing, perhaps, concern the implications for the choice of investment projects by a developing economy that is determined to grow as rapidly as possible. There are also intriguing theoretical questions concerning the nature of general equilibrium in an economy in which all firms select their investments in order to maximize their rates of growth. Preliminary explorations indicate that all such questions are too complicated to be dealt with in the confines of this paper.

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