Prove algebraically that for complex numbers,

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Interpret this result in terms of two-dimensional vectors. Prove that

$$|z - 1| < \left|\sqrt{z^2 - 1}\right| < |z + 1|, \qquad \text{for} \quad \Re(z) > 0.$$

Show that complex numbers have square roots and that the square roots are contained in the complex plane. What are the square roots of $i$?

Show that

(a) $\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2}\theta \sin^2\theta + \binom{n}{4} \cos^{n-4}\theta \sin^4\theta - \cdots.$
(b) $\sin n\theta = \binom{n}{1} \cos^{n-1}\theta \sin\theta - \binom{n}{3} \cos^{n-3}\theta \sin^3\theta + \cdots.$

*Note.* The quantities $\binom{n}{m}$ are binomial coefficients: $\binom{n}{m} = n!/[(n-m)!m!]$.

Find the analytic function

$$w(z) = u(x, y) + iv(x, y)$$

if (a) $u(x, y) = x^3 - 3xy^2$, (b) $v(x, y) = e^{-y} \sin x$.

If there is some common region in which $w_1 = u(x, y) + iv(x, y)$ and $w_2 = w_1^* = u(x, y) - iv(x, y)$ are both analytic, prove that $u(x, y)$ and $v(x, y)$ are constants.

The function $f(z) = u(x, y) + iv(x, y)$ is analytic. Show that $f^*(z^*)$ is also analytic.