Identify each number as real, complex, pure imaginary, *or* nonreal complex. *(More than one of these descriptions will apply.)*

7. -4 **8.** 0 **9.** $13i$ **10.** $-7i$ **11.** $5 + i$

12. $-6 - 2i$ **13.** π **14.** $\sqrt{24}$ **15.** $\sqrt{-25}$ **16.** $\sqrt{-36}$

Write each number as the product of a real number and i. **See Example 1.**

17. $\sqrt{-25}$ **18.** $\sqrt{-36}$ **19.** $\sqrt{-10}$ **20.** $\sqrt{-15}$

21. $\sqrt{-288}$ **22.** $\sqrt{-500}$ **23.** $-\sqrt{-18}$ **24.** $-\sqrt{-80}$

Solve each quadratic equation and express all nonreal complex solutions in terms of i. **See Examples 2 and 3.**

25. $x^2 = -16$ **26.** $x^2 = -36$

27. $x^2 + 12 = 0$ **28.** $x^2 + 48 = 0$

29. $3x^2 + 2 = -4x$ **30.** $2x^2 + 3x = -2$

31. $x^2 - 6x + 14 = 0$ **32.** $x^2 + 4x + 11 = 0$

33. $4(x^2 - x) = -7$ **34.** $3(3x^2 - 2x) = -7$

35. $x^2 + 1 = -x$ **36.** $x^2 + 2 = 2x$

Multiply or divide, as indicated. Simplify each answer. **See Example 4.**

37. $\sqrt{-13} \cdot \sqrt{-13}$ **38.** $\sqrt{-17} \cdot \sqrt{-17}$ **39.** $\sqrt{-3} \cdot \sqrt{-8}$

40. $\sqrt{-5} \cdot \sqrt{-15}$ **41.** $\frac{\sqrt{-30}}{\sqrt{-10}}$ **42.** $\frac{\sqrt{-70}}{\sqrt{-7}}$

43. $\frac{\sqrt{-24}}{\sqrt{8}}$ **44.** $\frac{\sqrt{-54}}{\sqrt{27}}$ **45.** $\frac{\sqrt{-10}}{\sqrt{-40}}$

46. $\frac{\sqrt{-8}}{\sqrt{-72}}$ **47.** $\frac{\sqrt{-6} \cdot \sqrt{-2}}{\sqrt{3}}$ **48.** $\frac{\sqrt{-12} \cdot \sqrt{-6}}{\sqrt{8}}$

Write each number in standard form a + bi. **See Example 5.**

49. $\frac{-6 - \sqrt{-24}}{2}$ **50.** $\frac{-9 - \sqrt{-18}}{3}$ **51.** $\frac{10 + \sqrt{-200}}{5}$

52. $\frac{20 + \sqrt{-8}}{2}$ **53.** $\frac{-3 + \sqrt{-18}}{24}$ **54.** $\frac{-5 + \sqrt{-50}}{10}$

Find each sum or difference. Write the answer in standard form. **See Example 6.**

55. $(3 + 2i) + (9 - 3i)$ **56.** $(4 - i) + (8 + 5i)$

57. $(-2 + 4i) - (-4 + 4i)$ **58.** $(-3 + 2i) - (-4 + 2i)$

59. $(2 - 5i) - (3 + 4i) - (-1 - 9i)$ **60.** $(-4 - i) - (2 + 3i) + (6 + 4i)$

61. $-i\sqrt{2} - 2 - (6 - 4i\sqrt{2}) - (5 - i\sqrt{2})$

62. $3\sqrt{7} - (4\sqrt{7} - i) - 4i + (-2\sqrt{7} + 5i)$

Find each product. Write the answer in standard form. **See Example 7.**

63. $(2 + i)(3 - 2i)$ **64.** $(-2 + 3i)(4 - 2i)$ **65.** $(2 + 4i)(-1 + 3i)$

66. $(1 + 3i)(2 - 5i)$ **67.** $(3 - 2i)^2$ **68.** $(2 + i)^2$

69. $(3 + i)(3 - i)$ **70.** $(5 + i)(5 - i)$ **71.** $(-2 - 3i)(-2 + 3i)$

72. $(6 - 4i)(6 + 4i)$ **73.** $(\sqrt{6} + i)(\sqrt{6} - i)$ **74.** $(\sqrt{2} - 4i)(\sqrt{2} + 4i)$

75. $i(3 - 4i)(3 + 4i)$ **76.** $i(2 + 7i)(2 - 7i)$ **77.** $3i(2 - i)^2$

78. $-5i(4 - 3i)^2$ **79.** $(2 + i)(2 - i)(4 + 3i)$ **80.** $(3 - i)(3 + i)(2 - 6i)$

Find each quotient. Write the answer in standard form a + bi. **See Example 8.**

81. $\frac{6 + 2i}{1 + 2i}$ **82.** $\frac{14 + 5i}{3 + 2i}$ **83.** $\frac{2 - i}{2 + i}$

84. $\frac{4 - 3i}{4 + 3i}$ **85.** $\frac{1 - 3i}{1 + i}$ **86.** $\frac{-3 + 4i}{2 - i}$

87. $\frac{-5}{i}$ **88.** $\frac{-6}{i}$ **89.** $\frac{8}{-i}$

90. $\frac{12}{-i}$ **91.** $\frac{2}{3i}$ **92.** $\frac{5}{9i}$

Simplify each power of i. **See Example 9.**

93. i^{25} **94.** i^{29} **95.** i^{22}

96. i^{26} **97.** i^{23} **98.** i^{27}

99. i^{32} **100.** i^{40} **101.** i^{-13}

102. i^{-14} **103.** $\frac{1}{i^{-11}}$ **104.** $\frac{1}{i^{-12}}$

105. Suppose that your friend, Kathy Strautz, tells you that she has discovered a method of simplifying a positive power of i. "Just divide the exponent by 2. Your answer is then the simplified form of i^2 raised to the quotient times i raised to the remainder." Explain why her method works.

106. Explain why the following method of simplifying i^{-42} works.

$$i^{-42} = \frac{1}{i^{42}} = \frac{1}{(i^2)^{21}} = \frac{1}{(-1)^{21}} = \frac{1}{-1} = -1$$

107. Show that $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ is a square root of i.

108. Show that $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ is a cube root of i.

109. Show that $-2 + i$ is a solution of the equation $x^2 + 4x + 5 = 0$.

110. Show that $-3 + 4i$ is a solution of the equation $x^2 + 6x + 25 = 0$.

(Modeling) Alternating Current *Complex numbers are used to describe current, I, voltage, E, and impedance, Z (the opposition to current). These three quantities are related by the equation E = IZ. Thus, if any two of these quantities are known, the third can be found. In each exercise, solve the equation E = IZ for the missing variable.*

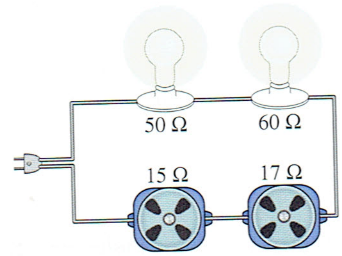
111. $I = 8 + 6i, \quad Z = 6 + 3i$ **112.** $I = 10 + 6i, \quad Z = 8 + 5i$

113. $I = 7 + 5i, \quad E = 28 + 54i$ **114.** $E = 35 + 55i, \quad Z = 6 + 4i$

(Modeling) Impedance ***Impedance*** *is a measure of the opposition to the flow of alternating electrical current found in common electrical outlets. It consists of two parts,* ***resistance*** *and* ***reactance.*** *Resistance occurs when a light bulb is turned on, while reactance is produced when electricity passes through a coil of wire like that found in electric motors. Impedance Z in ohms* (Ω) *can be expressed as a complex number, where the real part represents resistance and the imaginary part represents reactance.*

For example, if the resistive part is 3 ohms and the reactive part is 4 ohms, then the impedance could be described by the complex number $Z = 3 + 4i$. *In the series circuit shown in the figure, the total impedance will be the sum of the individual impedances.* (*Source:* Wilcox, G. and C. Hesselberth, *Electricity for Engineering Technology*, Allyn & Bacon.)

115. The circuit contains two light bulbs and two electric motors. Assuming that the light bulbs are pure resistive and the motors are pure reactive, find the total impedance in this circuit and express it in the form $Z = a + bi$.



116. The phase angle θ measures the phase difference between the voltage and the current in an electrical circuit. θ (in degrees) can be determined by the equation $\tan \theta = \frac{b}{a}$. Find θ for this circuit.

8.2 Trigonometric (Polar) Form of Complex Numbers

- The Complex Plane and Vector Representation
- Trigonometric (Polar) Form
- Converting between Rectangular and Trigonometric (Polar) Forms
- An Application of Complex Numbers to Fractals

The Complex Plane and Vector Representation

Unlike real numbers, complex numbers cannot be ordered. One way to organize and illustrate them is by using a graph.

To graph a complex number such as $2 - 3i$, we modify the familiar coordinate system by calling the horizontal axis the **real axis** and the vertical axis the **imaginary axis.** Then complex numbers can be graphed in this **complex plane,** as shown in **Figure 3.** ***Each complex number*** $a + bi$ ***determines a unique position vector with initial point*** $(0, 0)$ ***and terminal point*** (a, b).

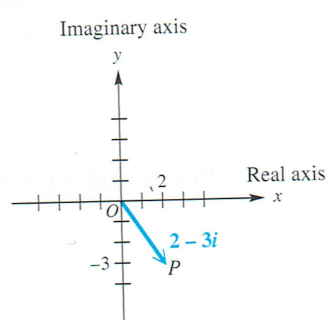


Figure 3

NOTE This geometric representation is the reason that $a + bi$ is called the **rectangular form** of a complex number. (*Rectangular form* is also called *standard form.*)

(b) For $z = 1 + 1i$, we have the following.

$$z^2 - 1 = (1 + i)^2 - 1 \qquad \text{Substitute for } z\text{: } 1 + 1i = 1 + i.$$

$$= (1 + 2i + i^2) - 1 \qquad \text{Square the binomial; } (x + y)^2 = x^2 + 2xy + y^2.$$

$$= -1 + 2i \qquad i^2 = -1$$

The absolute value is

$$\sqrt{(-1)^2 + 2^2} = \sqrt{5}.$$

Since $\sqrt{5}$ is greater than 2, the number $1 + 1i$ is not in the Julia set and $(1, 1)$ is not part of the graph.

✔ ***Now Try Exercise 63.***

8.2 Exercises

1. *Concept Check* The absolute value (or modulus) of a complex number represents the ________________ of the vector representing it in the complex plane.

2. *Concept Check* What is the geometric interpretation of the argument of a complex number?

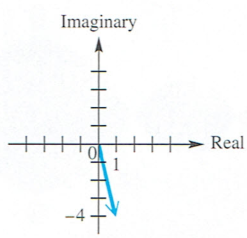
Graph each complex number. ***See Example 1.***

3. $-3 + 2i$ **4.** $6 - 5i$ **5.** $\sqrt{2} + \sqrt{2}i$ **6.** $2 - 2i\sqrt{3}$

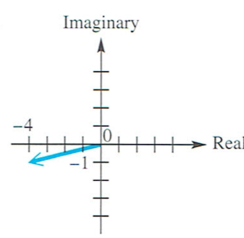
7. $-4i$ **8.** $3i$ **9.** -8 **10.** 2

Concept Check Give the rectangular form of the complex number shown.

11.



12.



Find the sum of each pair of complex numbers. In Exercises 13–16, graph both complex numbers and their resultant. ***See Example 1.***

13. $4 - 3i, -1 + 2i$ **14.** $2 + 3i, -4 - i$ **15.** $5 - 6i, -5 + 3i$

16. $7 - 3i, -4 + 3i$ **17.** $-3, 3i$ **18.** $6, -2i$

19. $-5 - 8i, -1$ **20.** $4 - 2i, 5$ **21.** $7 + 6i, 3i$

22. $2 + 6i, -2i$ **23.** $\frac{1}{2} + \frac{2}{3}i, \frac{2}{3} + \frac{1}{2}i$ **24.** $-\frac{1}{5} + \frac{2}{7}i, \frac{3}{7} - \frac{3}{4}i$

Write each complex number in rectangular form. ***See Example 2.***

25. $2(\cos 45° + i \sin 45°)$ **26.** $4(\cos 60° + i \sin 60°)$

27. $10(\cos 90° + i \sin 90°)$ **28.** $8(\cos 270° + i \sin 270°)$

29. $4(\cos 240° + i \sin 240°)$ **30.** $2(\cos 330° + i \sin 330°)$

31. $3 \text{ cis } 150°$ **32.** $2 \text{ cis } 30°$

33. $5 \text{ cis } 300°$

34. $6 \text{ cis } 135°$

35. $\sqrt{2} \text{ cis } 225°$

36. $\sqrt{3} \text{ cis } 315°$

37. $4(\cos(-30°) + i \sin(-30°))$

38. $\sqrt{2}(\cos(-60°) + i \sin(-60°))$

Write each complex number in trigonometric form $r(\cos \theta + i \sin \theta)$, with θ in the interval $[0°, 360°)$. ***See Example 3.***

39. $-3 - 3i\sqrt{3}$

40. $1 + i\sqrt{3}$

41. $\sqrt{3} - i$

42. $4\sqrt{3} + 4i$

43. $-5 - 5i$

44. $-2 + 2i$

45. $2 + 2i$

46. $4 + 4i$

47. $5i$

48. $-2i$

49. -4

50. 7

Perform each conversion, using a calculator to approximate answers as necessary. ***See Example 4.***

	Rectangular Form	Trigonometric Form
51.	$2 + 3i$	______
52.	______	$\cos 35° + i \sin 35°$
53.	______	$3(\cos 250° + i \sin 250°)$
54.	$-4 + i$	______
55.	$12i$	______
56.	______	$3 \text{ cis } 180°$
57.	$3 + 5i$	______
58.	______	$\text{cis } 110.5°$

Concept Check *The complex number z, where $z = x + yi$, can be graphed in the plane as (x, y). Describe the graphs of all complex numbers z satisfying the conditions in Exercises 59–62.*

59. The absolute value of z is 1.

60. The real and imaginary parts of z are equal.

61. The real part of z is 1.

62. The imaginary part of z is 1.

Julia Set *Refer to* ***Example 5*** *to solve Exercises 63 and 64.*

63. Is $z = -0.2i$ in the Julia set?

64. The graph of the Julia set in **Figure 11** appears to be symmetric with respect to both the x-axis and the y-axis. Complete the following to show that this is true.

- **(a)** Show that complex conjugates have the same absolute value.
- **(b)** Compute $z_1^2 - 1$ and $z_2^2 - 1$, where $z_1 = a + bi$ and $z_2 = a - bi$.
- **(c)** Discuss why if (a, b) is in the Julia set, then so is $(a, -b)$.
- **(d)** Conclude that the graph of the Julia set must be symmetric with respect to the x-axis.
- **(e)** Using a similar argument, show that the Julia set must also be symmetric with respect to the y-axis.

In Exercises 65 and 66, suppose $z = r(\cos \theta + i \sin \theta)$.

65. Use vectors to show that the conjugate of z is

$$r[\cos(360° - \theta) + i \sin(360° - \theta)], \quad \text{or} \quad r(\cos \theta - i \sin \theta).$$

66. Use vectors to show that

$$-z = r[\cos(\theta + \pi) + i \sin(\theta + \pi)].$$

Find each quotient and write it in rectangular form. In Exercises 19–24, first convert the numerator and the denominator to trigonometric form. **See Example 2.**

13. $\dfrac{4(\cos 150° + i \sin 150°)}{2(\cos 120° + i \sin 120°)}$

14. $\dfrac{24(\cos 150° + i \sin 150°)}{2(\cos 30° + i \sin 30°)}$

15. $\dfrac{10(\cos 50° + i \sin 50°)}{5(\cos 230° + i \sin 230°)}$

16. $\dfrac{12(\cos 23° + i \sin 23°)}{6(\cos 293° + i \sin 293°)}$

17. $\dfrac{3 \text{ cis } 305°}{9 \text{ cis } 65°}$

18. $\dfrac{16 \text{ cis } 310°}{8 \text{ cis } 70°}$

19. $\dfrac{8}{\sqrt{3} + i}$

20. $\dfrac{2i}{-1 - i\sqrt{3}}$

21. $\dfrac{-i}{1 + i}$

22. $\dfrac{1}{2 - 2i}$

23. $\dfrac{2\sqrt{6} - 2i\sqrt{2}}{\sqrt{2} - i\sqrt{6}}$

24. $\dfrac{-3\sqrt{2} + 3i\sqrt{6}}{\sqrt{6} + i\sqrt{2}}$

Use a calculator to perform the indicated operations. Give answers in rectangular form, expressing real and imaginary parts to four decimal places. **See Example 3.**

25. $[2.5(\cos 35° + i \sin 35°)][3.0(\cos 50° + i \sin 50°)]$

26. $[4.6(\cos 12° + i \sin 12°)][2.0(\cos 13° + i \sin 13°)]$

27. $(12 \text{ cis } 18.5°)(3 \text{ cis } 12.5°)$

28. $(4 \text{ cis } 19.25°)(7 \text{ cis } 41.75°)$

29. $\dfrac{45\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)}{22.5\left(\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5}\right)}$

30. $\dfrac{30\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)}{10\left(\cos \frac{\pi}{7} + i \sin \frac{\pi}{7}\right)}$

31. $\left[2 \text{ cis } \dfrac{5\pi}{9}\right]^2$

32. $\left[24.3 \text{ cis } \dfrac{7\pi}{12}\right]^2$

Relating Concepts

For individual or collaborative investigation *(Exercises 33–39)*

Consider the following complex numbers, and **work Exercises 33–39 in order.**

$$w = -1 + i \quad \text{and} \quad z = -1 - i$$

33. Multiply w and z using their rectangular forms and the FOIL method from **Section 8.1.** Leave the product in rectangular form.

34. Find the trigonometric forms of w and z.

35. Multiply w and z using their trigonometric forms and the method described in this section.

36. Use the result of **Exercise 35** to find the rectangular form of wz. How does this compare to your result in **Exercise 33?**

37. Find the quotient $\frac{w}{z}$ using their rectangular forms and multiplying both the numerator and the denominator by the conjugate of the denominator. Leave the quotient in rectangular form.

38. Use the trigonometric forms of w and z, found in **Exercise 34,** to divide w by z using the method described in this section.

39. Use the result of **Exercise 38** to find the rectangular form of $\frac{w}{z}$. How does this compare to your result in **Exercise 37?**

40. Note that $(r \text{ cis } \theta)^2 = (r \text{ cis } \theta)(r \text{ cis } \theta) = r^2 \text{ cis}(\theta + \theta) = r^2 \text{ cis } 2\theta$. Explain how we can square a complex number in trigonometric form. (In the next section, we will develop this idea more fully.)

41. Without actually performing the operations, state why the following products are the same.

$$[2(\cos 45° + i \sin 45°)] \cdot [5(\cos 90° + i \sin 90°)]$$

and $$[2[\cos(-315°) + i \sin(-315°)]] \cdot [5[\cos(-270°) + i \sin(-270°)]]$$

42. Show that $\frac{1}{z} = \frac{1}{r}(\cos \theta - i \sin \theta)$, where $z = r(\cos \theta + i \sin \theta)$.

(Modeling) Solve each problem.

43. ***Electrical Current*** The alternating current in an electric inductor is $I = \frac{E}{Z}$ amperes, where E is voltage and $Z = R + X_L i$ is impedance. If $E = 8(\cos 20° + i \sin 20°)$, $R = 6$, and $X_L = 3$, find the current. Give the answer in rectangular form, with real and imaginary parts to the nearest hundredth.

44. ***Electrical Current*** The current I in a circuit with voltage E, resistance R, capacitive reactance X_c, and inductive reactance X_L is

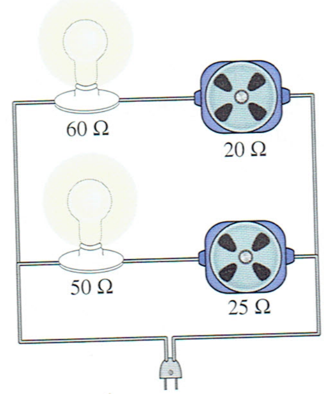
$$I = \frac{E}{R + (X_L - X_c)i}.$$

Find I if $E = 12(\cos 25° + i \sin 25°)$, $R = 3$, $X_L = 4$, and $X_c = 6$. Give the answer in rectangular form, with real and imaginary parts to the nearest tenth.

(Modeling) Impedance *In the parallel electrical circuit shown in the figure, the impedance Z can be calculated using the equation*

$$Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}},$$

where Z_1 and Z_2 are the impedances for the branches of the circuit.



45. If $Z_1 = 50 + 25i$ and $Z_2 = 60 + 20i$, calculate Z.

46. Determine the angle θ for the value of Z found in **Exercise 45.**

8.4 De Moivre's Theorem; Powers and Roots of Complex Numbers

- Powers of Complex Numbers (De Moivre's Theorem)
- Roots of Complex Numbers

Powers of Complex Numbers (De Moivre's Theorem) Because raising a number to a positive integer power is a repeated application of the product rule, it would seem likely that a theorem for finding powers of complex numbers exists. Consider the following.

$$[r(\cos \theta + i \sin \theta)]^2$$

$$= [r(\cos \theta + i \sin \theta)][r(\cos \theta + i \sin \theta)] \quad a^2 = a \cdot a$$

$$= r \cdot r[\cos(\theta + \theta) + i \sin(\theta + \theta)] \quad \text{Product theorem (Section 8.3)}$$

$$= r^2(\cos 2\theta + i \sin 2\theta) \quad \text{Multiply and add.}$$