The two problems are taken from this derivation of the inclusion-exclusion principle:

http://www.proofwiki.org/wiki/Inclusion-Exclusion_Principle#Induction_Hypothesis

Theorem

Let $\mathcal{S}$ be an algebra of sets.

Let $A_1, A_2, \dots, A_n$ be finite sets.

Let $f : \mathcal{S} \to \mathbb{R}$ be an additive function.

Then:

$$\begin{aligned} f\left(\bigcup_{i=1}^{n} A_i\right) = & \sum_{i=1}^{n} f(A_i) \\ & - \sum_{1 \le i < j \le n} f(A_i \cap A_j) \\ & + \sum_{1 \le i < j < k \le n} f(A_i \cap A_j \cap A_k) \\ & \cdots \\ & + (-1)^{n-1} f\left(\bigcap_{i=1}^{n} A_i\right) \end{aligned}$$

Corollary

Let $\mathcal{S}$ be an algebra of sets.

Let $A_1, A_2, \dots, A_n$ be finite sets which are pairwise disjoint.

Let $f : \mathcal{S} \to \mathbb{R}$ be an additive function.

Then:

$$f\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} f(A_i)$$

Proof

Proof by induction:

For all $n \in \mathbb{N}^*$, let $P(n)$ be the proposition:

$$\begin{aligned} f\left(\bigcup_{i=1}^{n} A_i\right) = & \sum_{i=1}^{n} f(A_i) \\ & - \sum_{1 \le i < j \le n} f(A_i \cap A_j) \\ & + \sum_{1 \le i < j < k \le n} f(A_i \cap A_j \cap A_k) \\ & \cdots \\ & + (-1)^{n-1} f\left(\bigcap_{i=1}^{n} A_i\right) \end{aligned}$$

$P(1)$ is true, as this just says $f(A_1) = f(A_1)$.

Basis for the Induction

$P(2)$ is the case:

$$f(A_1 \cup A_2) = f(A_1) + f(A_2) - f(A_1 \cap A_2)$$

which is the result Additive Function on Union of Sets.

This is our basis for the induction.

Induction Hypothesis

Now we need to show that, if $P(r)$ is true, where $r \ge 2$, then it logically follows that $P(r+1)$ is true.

So this is our induction hypothesis:

$$
\begin{aligned}
f\left(\bigcup_{i=1}^{r} A_i\right) = & \sum_{i=1}^{r} f(A_i) \\
& - \sum_{1 \le i < j \le r} f(A_i \cap A_j) \\
& + \sum_{1 \le i < j < k \le r} f(A_i \cap A_j \cap A_k) \\
& \cdots \\
& + (-1)^{r-1} f\left(\bigcap_{i=1}^{r} A_i\right)
\end{aligned}
$$

Then we need to show:

$$
\begin{aligned}
f\left(\bigcup_{i=1}^{r+1} A_i\right) = & \sum_{i=1}^{r+1} f(A_i) \\
& - \sum_{1 \le i < j \le r+1} f(A_i \cap A_j) \\
& + \sum_{1 \le i < j < k \le r+1} f(A_i \cap A_j \cap A_k) \\
& \cdots \\
& + (-1)^{r} f\left(\bigcap_{i=1}^{r+1} A_i\right)
\end{aligned}
$$

Induction Step

This is our induction step:

$$
\begin{aligned}
f\left(\bigcup_{i=1}^{r+1} A_i\right) &= f\left(\bigcup_{i=1}^{r} A_i \cup A_{r+1}\right) \\
&= f\left(\bigcup_{i=1}^{r} A_i\right) + f(A_{r+1}) - f\left(\bigcup_{i=1}^{r} A_i \cap A_{r+1}\right) \qquad \text{from the base case}
\end{aligned}
$$

Consider $f\left(\bigcup_{i=1}^{r} A_i \cap A_{r+1}\right)$.

By the fact that Intersection Distributes over Union, this can be written:

$$f\left(\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right)$$

To this, we can apply the induction hypothesis:

$$\begin{aligned} f\left(\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right) = & \sum_{i=1}^{r} f(A_i \cap A_{r+1}) \\ & - \sum_{1 \le i < j \le r} f(A_i \cap A_j \cap A_{r+1}) \\ & + \sum_{1 \le i < j < k \le r} f(A_i \cap A_j \cap A_k \cap A_{r+1}) \\ & \cdots \\ & + (-1)^{r-1} f\left(\bigcap_{i=1}^{r} A_i \cap A_{r+1}\right) \end{aligned}$$

Why can they apply the induction hypothesis to the part here as well? Have they not changed the conditions by using $A_i \cap A_{r+1}$?

At the same time, we have the expansion of the term $f\left(\bigcup_{i=1}^{r} A_i\right)$ to take into account.

So we can consider the general term of s intersections in the expansion of $f\left(\bigcup_{i=1}^{r+1} A_i\right)$:

$$(-1)^{s-1} \sum_{\substack{i \in I \\ |I| = s}} f\left(\bigcap_{i \in I} A_i\right) - (-1)^{s-2} \sum_{\substack{i \in J \\ |J| = s-1}} f\left(\bigcap_{i \in J} A_i \cap A_{r+1}\right)$$

where:

- I ranges over all sets of s elements out of $[1 \,.\,.\, r]$
- J ranges over all sets of $s - 1$ elements out of $[1 \,.\,.\, r]$
- $1 \le s \le r$

Where do we get to take into account the expansion of

$$f\left(\bigcup_{i=1}^{r} A_i\right)$$

? What does it mean? And how do they end up with the part

$$(-1)^{s-1} \sum_{\substack{I \subseteq [1 \,.\,.\, r] \\ |I| = s}} f\left(\bigcap_{i \in I} A_i\right) - (-1)^{s-2} \sum_{\substack{J \subseteq [1 \,.\,.\, r] \\ |J| = s-1}} f\left(\bigcap_{i \in J} A_i \cap A_{r+1}\right)$$

where:

- I ranges over all sets of s elements out of $[1 \,.\,.\, r]$
- J ranges over all sets of $s - 1$ elements out of $[1 \,.\,.\, r]$
- $1 \le s \le r$