From Hildebrand. Adv. Calc. for Applications. 2nd Edition PrenticeHall 1976.

p.612.

98. It is required to show that

$$I_1 \equiv \int_0^\infty \frac{e^{-xt}}{x^2+1}\,dx = \int_0^\infty \frac{\sin rt}{r+1}\,dr \equiv I_2 \qquad (t > 0).$$

(a) Show that

$$I_1 = \frac{1}{2i}\left(\int_0^\infty \frac{e^{-xt}}{x-i}\,dx - \int_0^\infty \frac{e^{-xt}}{x+i}\,dx\right).$$

(b) By considering the limit of the integral of the function

$$f(z) = \frac{e^{-zt}}{z-i}$$

around the sector $0 \leqq r \leqq R$, $-\pi/2 \leqq \theta \leqq 0$, as $R \to \infty$, show that

$$\int_0^\infty \frac{e^{-xt}}{x-i}\,dx = \int_0^\infty \frac{e^{irt}}{r+1}\,dr$$

when $t > 0$. (Use Theorem II.4.) — See p. 589–591 →

(c) In a similar way, show that

$$\int_0^\infty \frac{e^{-xt}}{x+i}\,dx = \int_0^\infty \frac{e^{-irt}}{r+1}\,dr,$$

when $t > 0$, and hence complete the proof.

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10.14. Theorems on Limiting Contours. In many applications of contour integration it is necessary to evaluate the limit of the result of integrating a function of a complex variable along an arc of a circle as the radius of that circle either increases without limit or tends to zero. In this section we collect and establish certain general results of frequent application. First, however, it is convenient to introduce a useful definition.

If, along a circular arc C_r of radius r, we have $|f(z)| \leqq K_r$, where K_r is a bound depending only on r and hence independent of *angular* position on C_r, and if $K_r \to 0$ as $r \to \infty$ (or $r \to 0$), then we will say that $f(z)$ *tends to zero uniformly* on C_r as $r \to \infty$ (or $r \to 0$). Thus, for example, if C_r is a circular arc with center at the origin and $f(z) = z/(z^2+1)$, we have

$$|f(z)| = \frac{|z|}{|z^2+1|} \leqq \frac{|z|}{|z|^2-1} = \frac{r}{r^2-1} \qquad (r > 1)$$

on C_r, if use is made of the basic inequality (10). Hence, if we then take $K_r = r/(r^2-1)$, we conclude that here $f(z)$ tends to zero uniformly on C_r as $r \to \infty$. Also, we may take $K_r = r/(1-r^2)$ when $r < 1$ to show that the same is true when $r \to 0$.

In particular, any *rational function* (ratio of polynomials) whose denominator is of higher degree than the numerator tends uniformly to zero on any C_r as $r \to \infty$. This follows from the fact that then $|z|\,|f(z)|$ tends to a limit (which may be zero) as $|z| = r \to \infty$, and hence is bounded by some constant k when r is large (say, $r \geqq r_0$), so that we may take $K_r = k/r$ (when $r \geqq r_0$).

The following theorems now may be stated:

Theorem I. If, on a circular arc C_R with radius R and center at the origin, $zf(z) \to 0$ uniformly as $R \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

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Theorem II. Suppose that, on a circular arc C_R with radius R and center at the origin, $f(z) \longrightarrow 0$ uniformly as $R \longrightarrow \infty$. Then:

1. $$\lim_{R\to\infty} \int_{C_R} e^{imz} f(z)\, dz = 0 \qquad (m > 0)$$

if C_R is in the first and/or second quadrants.†

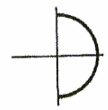
2. $$\lim_{R\to\infty} \int_{C_R} e^{-imz} f(z)\, dz = 0 \qquad (m > 0)$$

if C_R is in the third and/or fourth quadrants.

3. $$\lim_{R\to\infty} \int_{C_R} e^{mz} f(z)\, dz = 0 \qquad (m > 0)$$

if C_R is in the second and/or third quadrants.

4. $$\lim_{R\to\infty} \int_{C_R} e^{-mz} f(z)\, dz = 0 \qquad (m > 0)$$



if C_R is in the first and/or fourth quadrants.

Theorem III. If, on a circular arc C_ρ with radius ρ and center at $z = a$, $(z - a)f(z) \longrightarrow 0$ uniformly as $\rho \longrightarrow 0$, then

$$\lim_{\rho\to 0} \int_{C_\rho} f(z)\, dz = 0.$$

Theorem IV. Suppose that $f(z)$ has a simple pole at $z = a$, with residue Res (a). Then, if C_ρ is a circular arc with radius ρ and center at $z = a$, intercepting an angle α at $z = a$, there follows

$$\lim_{\rho\to 0} \int_{C_\rho} f(z)\, dz = \alpha i \operatorname{Res}(a),$$

where α is positive if the integration is carried out in the counterclockwise direction, and negative otherwise.

The proof of Theorem I follows from the fact that if $|zf(z)| \leqq K_R$, then $|f(z)| \leqq K_R/R$. Since the length of C_R is $|\alpha|R$, where α is the subtended angle, Equation (80) gives

$$\left|\int_{C_R} f(z)\, dz\right| \leqq \frac{K_R}{R} \cdot |\alpha|R = |\alpha|K_R \longrightarrow 0 \qquad (R \longrightarrow \infty). \qquad ∎$$

The proof of Theorem II is somewhat more complicated. To prove part 1, we use the relation

$$\left|\int_{C_R} e^{imz} f(z)\, dz\right| \leqq \int_{C_R} |e^{imz}|\,|f(z)|\,|dz|.$$

†This result is known as *Jordan's lemma.*

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But on C_R we have $|dz| = R\, d\theta, |f(z)| \leqq K_R$, and $|e^{imz}| = e^{-mR\sin\theta}$, acc to (150a). Hence there follows

$$|I_R| \equiv \left|\int_{C_R} e^{imz} f(z)\, dz\right| \leqq RK_R \int_{\theta_0}^{\theta_1} e^{-mR\sin\theta}\, d\theta,$$

where $0 \leqq \theta_0 < \theta_1 \leqq \pi$. Since the last integrand is positive, the right member is not decreased if we take $\theta_0 = 0$ and $\theta_1 = \pi$. Hence we have

$$|I_R| \leqq RK_R \int_0^{\pi} e^{-mR\sin\theta}\, d\theta = 2RK_R \int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta.$$

This integral cannot be evaluated in terms of elementary functions of R. ever, in the range $0 \leqq \theta \leqq \pi/2$ the truth of the relation

$$\sin\theta \geqq \frac{2}{\pi}\theta$$

is easily realized by comparing the graphs of $y = \sin x$ and $y = 2x/\pi$. we have also, from (160),

$$|I_R| \leqq 2RK_R \int_0^{\pi/2} e^{-2mR\theta/\pi}\, d\theta = \frac{\pi}{m} K_R(1 - e^{-mR}),$$

and hence, if $m > 0$, I_R tends to zero with K_R as $R \longrightarrow \infty$, as was to be s

The other three parts of Theorem II are established by completel ogous methods.

To prove Theorem III we notice that the integrand is not greate K_ρ/ρ in absolute value and the length of the path is $|\alpha|\rho$, where α is th tended angle. Hence the integral tends to zero with K_ρ.

To establish Theorem IV, we notice that if $f(z)$ has a simple pole at we can write

$$f(z) = \frac{\operatorname{Res}(a)}{z - a} + \varphi(z),$$

where $\varphi(z)$ is *analytic*, and hence bounded, in the neighborhood of Hence we have

$$\int_{C_\rho} f(z)\, dz = \int_{C_\rho} \frac{\operatorname{Res}(a)}{z - a}\, dz + \int_{C_\rho} \varphi(z)\, dz.$$

On C_ρ we can write $z = a + \rho e^{i\theta}$, where θ varies from an initial valu $\theta_0 + \alpha$. Hence the first integral on the right becomes

$$\operatorname{Res}(a) \int_{\theta_0}^{\theta_0+\alpha} \frac{\rho i e^{i\theta}\, d\theta}{\rho e^{i\theta}} = i \operatorname{Res}(a) \int_{\theta_0}^{\theta_0+\alpha} d\theta = \alpha i \operatorname{Res}(a).$$

The second integral on the right tends to zero with ρ, in consequen Theorem III, establishing the desired result.