(i) Not everything with indices is a tensor. If ϕ and A^μ transform as a scalar and a contravariant vector under *general* coordinate transformations, respectively, then show that

$$C_\mu := \frac{\partial \phi}{\partial x^\mu}$$

is a tensor, but that

$$T^\mu{}_\nu := \frac{\partial A^\mu}{\partial x^\nu}$$

is not. [Hint: apply the transformation laws for ϕ, A^μ and ∂_ν to determine how the RHS transforms in each case.]

(ii) Let $g^{\mu\nu}$ be a symmetric tensor function under general coordinate transformations, i.e., $g^{\mu\nu} = g^{\nu\mu}$, and let $h_{\mu\nu}$ be a tensor that satisfies

$$g^{\mu\nu} h_{\nu\lambda} = \delta^\mu{}_\lambda$$

in a particular set of coordinates.

(a) Is the above relation satisfied in all sets of coordinates? Why or why not? [Hint: see Eq. (2.4) of the notes.]

(b) By multiplying each side of the above relation by $h_{\mu\alpha}$, show that $h_{\alpha\lambda} = h_{\lambda\alpha}$, i.e., that $h_{\mu\nu}$ is also symmetric.

(c) By differentiating each side of the above relation with respect to x^α, show that

$$\frac{\partial h_{\beta\nu}}{\partial x^\alpha} = -h_{\beta\mu} h_{\nu\lambda} \frac{\partial g^{\mu\lambda}}{\partial x^\alpha}.$$

[Warning: never use the same dummy index more than twice in a given expression. Otherwise the summation convention becomes inconsistent. For example, $(A^\mu B_\mu)^2 = A^\mu B_\mu A^\nu B_\nu \neq A^\mu B_\mu A^\mu B_\mu$, if the latter expression is interpreted as implying a sum over μ.]