Multivariate Calculus and Optimization

Kuntal Banerjee*

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1 Multivariate Calculus

1.1 Functions on Two Variables

We have studied functions of single variables, described detailed procedures of graphing such functions. In these notes we will provide a brief summary on multivariate calculus.

Multivariate functions are functions that depend on more than one-variable. Assume that you drive to school. At what time you reach school depends on two things: (a) when you left home (h) and (b) how much traffic congestion or other circumstances there is on the way to school (c). So if you were to establish a relation between the time you make it to school (denote that by the variable t) and the items (a) and (b), then that relation would be a multivariate function. If precise nature of this dependence is captured through the function f, then $t = f(h, c)$. We describe next a precise way of describing such a function.

Consider function f defined on the set of "ordered pairs" of real numbers, that is set of pairs of real numbers (x_1, x_2) with $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. We will denote the set of ordered pairs of real numbers by $\mathbb{R}^2$. An example of such a function is $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x_1, x_2) = x_1 + (x_2)^2$. The domain of the function is $\mathbb{R}^2$ and it takes values on the real line. So, the value of the function at the point $(1, -1)$ is given by $f(1, -1) = 1 + (-1)^2 = 2$. It should be noted that writing ordered

*Address: Barry Kaye School of Business, Room KH 105, Florida Atlantic University, Boca Raton FL 33431, USA. Email: kbanerje@fau.edu or kuntal.banerjee@gmail.com

pairs as (x_1, x_2) should not be confused with the notation we used in class to describe an open interval (Recall that for any $a, b \in \mathbb{R}$ with $a < b$, the open interval (a, b) is the set of all the real numbers x such that $a < x < b$). It should be clear from the context of the definition and the algebraic formula associated with the multivariate function that we are indeed talking of ordered pairs when describing the function f, and not the open interval, which is a subset of $\mathbb{R}$.

The concept of a *partial derivative* of a multivariate function captures the rate of change of the value of a function with respect to a change in a particular variable keeping the other variables fixed. A function that has partial derivatives with respect to all its variables is called a *differentiable* multivariate function. Consider the function $f(x_1, x_2) = x_1 + (x_2)^2$, we will denote the partial derivative of f with respect to variables 1 and 2 by $(\partial f/\partial x_1)$ and $(\partial f/\partial x_2)$ respectively. In your book the discussion of multivariate functions uses the notation $f(x, y)$ instead of $f(x_1, x_2)$, this notation is not followed here, however they are one and the same thing.

An alternative convenient way of writing will be employed in this note, $(\partial f/\partial x_1)$ will be written as $f_1(x_1, x_2)$ and $(\partial f/\partial x_2)$ will be written as $f_2(x_1, x_2)$. We study the partial derivatives through the following examples.

Intuitively, partial derivatives determine the slope of the function along a particular variable. It tells us how the function changes if a particular variable changes, so in the context of the example of the time when you reach school, it is likely that keeping the level of traffic congestion fixed, you are going to reach school later if you left home 5 minutes later. Similarly, if on two separate days you left home at 10 am and on the first day you experienced less traffic than the second, then you will reach school later in the second day as compared to the first.

Example 1: $f(x_1, x_2) = x_1 + (x_2)^2$, then $f_1(x_1, x_2) = 1$ and $f_2(x_1, x_2) = 2x_2$. We can also evaluate each of the partial derivatives at different points in the domain of the function, for instance the value of the partial derivative of f with respect to x_1 at $(0, 0)$ is given by $f_1(0, 0) = 1$ and the value of the partial derivative of f with respect to x_2 at $(0, 0)$ is given by $f_2(0, 0) = 0$.

Example 2: Consider the following function $f(x_1, x_2) = [x_1/(x_1 - x_2)]$ where the domain of f is the set $X = \{(x_1, x_2) : x_1 \geq 2 \text{ and } 0 \leq x_2 \leq 1\}$.

Exercise 1: (a) For any $(x_1, x_2) \in X$, what are the prtial derivatives of f with respect to x_1 and x_2, so what are the algebraic expressions for $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ (b) What are the values of the partial derivatives at the point $(3, 1/2) \in X$?

Example 3: Let $f(x_1, x_2) = x_1 x_2^2$ be defined on the set $Y = \{(x_1, x_2) : x_1 > 0 \text{ and } x_2 \in \mathbb{R}\}$.

Exercise 2: Find the expression for the partial derivative of f with respect to variables x_1 and x_2 and evaluate the derivatives at the point $(1, -2)$.

Example 4: Let $f(x_1, x_2) = \sqrt{(x_1 - 1)(2 - x_2)}$ be defined on the set $Z = \{(x_1, x_2) : x_1 > 1$ and $x_2 < 2\}$.

Exercise 3: Find the expression for the partial derivative of f with respect to variables x_1 and x_2 and evaluate the derivatives at the point $(2, -2)$.

In each of the above examples we see that the partial derivatives can be written as functions of the variables (x_1, x_2). So analogous to second order derivatives of single variable functions we can obtain second order partial derivatives. Given a differentiable function $f : X \to \mathbb{R}$ with $X \subseteq \mathbb{R}^2$ with partial derivatives $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$, we can take partial differential of $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ with respect to the variables x_1 and x_2 again. More precisely associate with $f_1(x_1, x_2)$ the partial derivatives $\partial[f_1(x_1, x_2)]/\partial x_1$ and $\partial[f_1(x_1, x_2)]/\partial x_2$ and we will denote them by $f_{11}(x_1, x_2)$ and $f_{12}(x_1, x_2)$. Similarly, associate with $f_2(x_1, x_2)$ the partial derivatives $\partial[f_2(x_1, x_2)]/\partial x_1$ and $\partial[f_2(x_1, x_2)]/\partial x_2$ and we will denote them by $f_{21}(x_1, x_2)$ and $f_{22}(x_1, x_2)$. So unlike a single variable function, there are 4 second order partial derivatives associated with a function which is defined on two variables. In example 1, $f_{11}(x_1, x_2) = 0$ and $f_{12}(x_1, x_2) = 0$ (note that is true because f_1 is a constant function). However, $f_{21}(x_1, x_2) = \partial[f_2(x_1, x_2)]/\partial x_1 = \partial[2x_2]/\partial x_1 = 0$ and $f_{22}(x_1, x_2) = \partial[f_2(x_1, x_2)]/\partial x_2 = \partial[2x_2]/\partial x_2 = 2$.

Exercise 4: Obtain algebraic expressions for $f_{11}(x_1, x_2)$, $f_{22}(x_1, x_2)$, $f_{21}(x_1, x_2)$ and $f_{22}(x_1, x_2)$ for the functions defined in Examples 2,3 and 4. Is $f_{12}(x_1, x_2) = f_{21}(x_1, x_2)$ in each case?

1.2 Concave Functions

Recall that for single variable functions we were concerned not only with the slope but also the curvature. Analogous tot he second derivative of a single variable function is the concept of the second order partial derivatives. These provide information on the curvature of the multivariate function f. We will be interested in a particular characterization of curvature, namely that of concave functions. The next result gives conditions on the second order partial derivatives guaranteeing that a function is concave.

Theorem 1 *Suppose $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}^2$ is a function with first and second partial derivatives (with respect to both variables) on A. f is concave on A if and only if*

$$\left.\begin{array}{c} f_{11}(x_1, x_2) \leq 0 \\ f_{22}(x_1, x_2) \leq 0 \\ (f_{11}(x_1, x_2))(f_{22}(x_1, x_2)) \geq [f_{12}(x_1, x_2)]^2 \end{array}\right\} (C)$$

for all $(x_1, x_2) \in A$.

It is useful to know a special case of Theorem 1 as we encounter a lot of functions whose domain is $\mathbb{R}^2_+$.

Theorem 2 *Suppose $f : A \to \mathbb{R}$ with $A = \mathbb{R}^2_+$ is a function with first and second partial derivatives (with respect to both variables) on $\mathbb{R}^2_+$. f is concave on $\mathbb{R}^2_+$ if and only if*

$$\left.\begin{array}{c} f_{11}(x_1, x_2) \leq 0 \\ f_{22}(x_1, x_2) \leq 0 \\ (f_{11}(x_1, x_2))(f_{22}(x_1, x_2)) \geq [f_{12}(x_1, x_2)]^2 \end{array}\right\} (C)$$

for all $(x_1, x_2) \in \mathbb{R}^2_{++}$.

Note that you only have to check that condition (C) holds for $(x_1, x_2) \in \mathbb{R}^2_{++}$, so the verification can be restricted to ordered pairs where each component is strictly positive,even when the function is defined on $\mathbb{R}^2_+$.

Let us demonstrate the concavity of $f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$ on the set $A = \{(x_1, x_2) : x_1 > 0$ and $x_2 > 0\}$. The first order partial derivatives of this function are given by $f_1(x_1, x_2) = 1/(2\sqrt{x_1})$ and $f_2(x_1, x_2) = 1/(2\sqrt{x_2})$ for $(x_1, x_2) \in A$. From each of the first order derivatives it is clear that, $f_{12}(x_1, x_2) = f_{21}(x_1, x_2) = 0$ and $f_{11}(x_1, x_2) = -(1/4)(x_1)^{-\frac{3}{2}}$, $f_{22}(x_1, x_2) = -(1/4)(x_2)^{-\frac{3}{2}}$for $(x_1, x_2) \in A$. Observe that for $(x_1, x_2) \in A$, we must have $f_{11}(x_1, x_2) = -(1/4)(x_1)^{-\frac{3}{2}} < 0$ (since $x_1 > 0$) and $f_{22}(x_1, x_2) = -(1/4)(x_2)^{-\frac{3}{2}} < 0$ (since $x_2 > 0$) verifying the first two conditions in (C). To check that the third condition in (C) holds note that $(f_{11}(x_1, x_2))(f_{22}(x_1, x_2)) > 0 = [f_{12}(x_1, x_2)]^2$ (the first inequality is true because both $D_{11}f$ and $D_{22}f$ were shown to be negative implying that their product is positive; the equality in the expression follows from noting that $f_{12}(x_1, x_2) = 0$). So, we have verified all three conditions needed to show that f is concave on A.

Exercise 5: Show that the function $f(x_1, x_2) = \sqrt{x_1 x_2}$ on the set $A = \{(x_1, x_2) : x_1 > 0$ and $x_2 > 0\}$ is concave on A.

1.3 Homogeneous Functions

Let A be a subset of $\mathbb{R}^2$ and $f : A \to \mathbb{R}$. We will say that f is homogeneous of degree r if

$$f(tx_1, tx_2) = t^r f(x_1, x_2) \text{ for } t > 0 \text{ and } (x_1, x_2) \in A.$$

This class of multivariate functions have a special role in economics of production. We present a result for homogeneous functions that will be useful in setting up production theory. Let us see one example. Let $f : \mathbb{R}^2_{++} \to \mathbb{R}_+$ be given by $f(x_1, x_2) = \sqrt{x_1 x_2}$. We can verify that f is homogeneous of degree 1. To see this take any $t > 0$ and $(x_1, x_2) \in \mathbb{R}^2_{++}$ to get $f(tx_1, tx_2) = \sqrt{(tx_1)(tx_2)} = \sqrt{t^2(x_1 x_2)} = t\sqrt{x_1 x_2} = tf(x_1, x_2)$ implying that f is homogeneous of degree 1.

Theorem 3 *Let A be a subset of $\mathbb{R}^2$ and $f : A \to \mathbb{R}$. Assume that f has continuous first order partial derivatives and homogeneous of degree r, then for any $(x_1, x_2) \in A$ we have*

$$x_1 f_1(x_1, x_2) + x_2 f_2(x_1, x_2) = rf(x_1, x_2).$$

Theorem 3 says how the f can be written as a sum of partial derivatives in the special case that f is homogeneous of degree r. With regards to the example discussed above $f(x_1, x_2) = \sqrt{x_1 x_2}$, we can verify that the conclusion of Theorem 3 is correct. Since $f_1(x_1, x_2) = (1/2)(x_2/x_1)^{1/2}$ and $f_2(x_1, x_2) = (1/2)(x_1/x_2)^{1/2}$ we get

$$\begin{aligned} x_1 f_1(x_1, x_2) + x_2 f_2(x_1, x_2) &= (1/2)x_1((x_2/x_1)^{1/2}) + (1/2)x_2((x_1/x_2)^{1/2}) \\ &= (1/2)[(x_1)^{1/2}(x_2)^{1/2} + (x_1)^{1/2}(x_2)^{1/2}] \\ &= (1/2)[2\sqrt{x_1 x_2}] \\ &= f(x_1, x_2). \end{aligned}$$

As shown before f is homogeneous of degree 1 and hence the analysis above validates Theorem 3 for the case when $f(x_1, x_2) = \sqrt{x_1 x_2}$.

Theorem 4 *Let A be a subset of $\mathbb{R}^2$ and $f : A \to \mathbb{R}$. Assume that f has continuous first order partial derivatives and homogeneous of degree r, then the partial derivatives $f_1 : A \to \mathbb{R}$ and $f_2 : A \to \mathbb{R}$ are homogeneous of degree $r - 1$.*

Theorem 4 says that if f is homogeneous of degree r, then f_1 and f_2 must be homogeneous of degree $(r-1)$.

Exercise 6: Let α be a real number with $0 < \alpha < 1$. Define the function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ by $F(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$. Verify the following:

(i) F is homogeneous of degree 1 on $\mathbb{R}^2_+$.

(ii) $F_1 : \mathbb{R}^2_{++} \to \mathbb{R}_+$ and $F_2 : \mathbb{R}^2_{++} \to \mathbb{R}_+$ the first order partial derivatives of F when looked at as functions from $\mathbb{R}^2_{++}$ to $\mathbb{R}_+$ are each functions of homogeneous of degree 0.

(iii) Show that the conclusion of Theorem 3 is also valid, i.e., $x_1 F_1(x_1, x_2) + x_2 F_2(x_1, x_2) = F(x_1, x_2)$ for any $(x_1, x_2) \in \mathbb{R}^2_{++}$.

(iv) Show that F is concave on $\mathbb{R}^2_+$.

Exercise 7: Let $F : B \to \mathbb{R}$ be the function

$$F(x_1, x_2) = x_1/[1 - (x_2 + 1)^2] \text{ for } (x_1, x_2) \in B$$

where $B = \{(x_1, x_2) : x_1 \geq 0 \text{ and } x_2 > -2\}$.

(i) Is the function F well defined? Qualify your answer with a mathematical argument [the phrase "well defined" means that is $F(x_1, x_2)$ a real number for all points (x_1, x_2) is the set B].

(ii) Give expressions for $F_1(x_1, x_2)$ and $F_2(x_1, x_2)$.