Examples of Exponential and Logarithmic Functions. Excercises to follow.

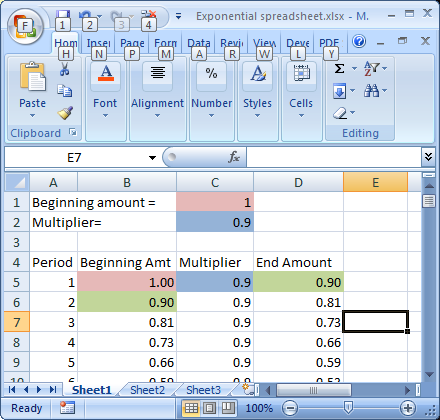
Explanation:

Start by writing the beginning amount into cell C1 (pink). The program automatically copies that number to cell B5 (pink), which is the beginning amount for the first period.

Type the multiplier into cell C3 (blue). The program automatically copies it to C5 (blue).

The End Amount for the period, in cell D5 (green), is calculated as A0(m), or B5\*C5. This amount is copied to the Beginning Amount of the next period, cell B6 (green).

The worksheet is set up for 100 periods.



**Exercises:**

1. Use the worksheet. Enter a beginning amount A0 = 10. Enter a multiplier m= 0.95. What is the End Amount at the end of the 10th period, A10 = ?

2. Check your answer in Exercise 1 by calculating A10=(A0)(0.95)10, using exponents. *(Any scientific calculator, including some costing less than $20, will evaluate this expression. Alternatively, you can find a scientific calculator app online.)*

3. Assume that A0=10 above refers to 10 grams of Mysterium (My). How many grams of My will left after 100 years?

4. Obviously, there are a very large number of atoms in any sample, and the process will come to an end when the very last atom decays. But let's assume there's an infinite number of atoms in the sample, which is a reasonable assumption for any *practical* purpose. If that assumption were true, then how long would it take for a sample of My to disappear *completely*?

To appreciate the behavior of An over time, plot An versus n, where values of n are on the X-axis and values of Y are on the Y-axis. Use the Excel Insert charts function, and choose the scatterplot option. Consult Excel Help if necessary.

The equation we've been working with,

(1)

has four variables. If we're given any three of them, we can solve for the fourth. In the form of the equation written above, we're given A0, m, and n, and can solve for An.

We can solve for any one of the four variables, given the other three. Using simple algebra, we can solve for the original amount, given the current amount, the multiplier, and the elapsed time. The equation (which the student should verify) is:

(2) 

Again using simple algebra, we can solve for the multiplier, given the original amount, the current amount, and the elapsed time. The equation is:

(3) 

Finally, we can solve for the elapsed time, n, given the original amount, the current amount, and the multiplier. This requires a bit more than simple algebra. We have to use the reciprocal relationship between exponential and logarithmic functions.

We begin with the original equation.



and take the log of both sides.



Since log(ab)=log(a)+log(b), we can rewrite the right side as follows.



Rearranging terms,



Now since log(xy)=ylog(x), the left side becomes



Dividing both sides by log[(m)], we arrive at the final equation

(4) 

**Exercises (Continued):**

5. Set up the spreadsheet with A0=100 and m= 0.99. At the end of year 20 (n=20), what is the end amount (A20 = ?)

In problems 6, 7 and 8, show your work *in detail.* The results of your calculations won't exactly match the worksheet results, owing to rounding error, but they should be close.

6. Verify the result by calculating A0, using equation (2) above.

7. Verify the result by calculating m, using equation (3) above.

8. Verify the result by calculating n, using equation (4) above.

9. A radioactive nuclide is commonly described in term of its "half-life." The half-life is the length of time it takes for half the amount of the nuclide to decay. If the decay product is not radioactive, then the radioactivity emitted by the sample is only half of what it was before.

Let's use our spreadsheet to estimate a half-life. The spreadsheet should be set up as before, with A0=100 and m= 0.99. Look at the column of An values, and find the number of years it takes for the amount of material to decrease from 100 to 50.

Calculate the half-life[[1]](#endnote-1), using equation (4). Again, show your work. (Note: There will substantial rounding error, owing to the small value of log(0.99).)

Example 2: Compound Interest

Interestingly, radioactive decay is mathematically related to compound interest. In the first process, a fixed amount of radioactive material gets smaller over time. In the second process, a fixed amount of money left on deposit gets larger over time.

Different nomenclature is used. Instead of A0, we write PV (for Present Value). Instead of An, we write FV (for Future Value). The number of compounding periods, usually years, is still written n. The interest earned per compounding period, written as a decimal, is the letter i; e.g., 5% =0.05.

Compare equation (1) above with the following compound interest equation;



Example: What's the future value of $100 invested at 5% interest, compounded annually, at the end of the first year (n=1)?



That's simple enough; we could probably have done that in our heads. But how much would $100 be worth, if left on deposit at 5% simple interest, for 20 years?



Let's check that using the worksheet. For the Beginning Amount, enter 100. For the Multiplier, enter 1.05. The future value -- that is, the End Amount -- at the end of 20 years is $265.33. The difference is due to rounding error, somewhere; either in our calculation, or in the worksheet's calculation.

You may have heard your Credit Union advertise, "Interest compounded DAILY on savings accounts!" That means the future value is updated on a daily basis. Instead of n=1 per year, n=365 per year. But you're certainly not going to get 5% per day. The interest per compounding period (one day) is the annual rate, divided by the number of days in a year. So an annual rate of 5% becomes a daily rate of (5/365)%, or 0.000137.

Let's see what the difference is, if our $100 investment at 5% per annum is compounded daily for one year.



Compare that with $105.00, calculated above. Big deal, huh? Of course, the difference becomes more attractive as the money is left on deposit for longer periods of time.

The other equations we derived for radioactive decay can be used, with suitable modifications, to solve for present value, interest, and number of compounding periods. But rather than show worked-out examples, we'll set them as --

**Exercises (Continued).**

10. A sum of money left on deposit, at interest compounded annually, doubles in value in exactly 10 years. What is the annual interest rate paid? Show your work.

11. At the time this SLP was written, T-notes were paying between 1% and 2% per annum. How many years would it take for $1000, left on deposit at 2% compounded annually, to double in value? Again, show your work.

The result clearly indicates the importance of the long game, particularly when you're playing with very safe, very low-paying securities.

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If you're a business major, you'll be taking courses in finance. Finance courses have a lot more to say about investments; sinking funds, periodic contributions, and the like. As you'll discover, many of the equations involve either exponents or logarithms. Something to look forward to!

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1. Physicists actually use a different, more accurate procedure for calculating the half-life. [↑](#endnote-ref-1)