Proof. If $R = A \cup B$ where $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, we must show that either A or B contains a limit point of the other. Let

$$E = \{\mathbf{x} \in D : f(\mathbf{x}) \in A\} = f^{-1}(A)$$
$$F = \{\mathbf{x} \in D : f(\mathbf{x}) \in B\} = f^{-1}(B)$$

Note that since $A \neq \emptyset$, $E \neq \emptyset$ and similarly $B \neq \emptyset$, $F \neq \emptyset$. Also $A \cap B = \emptyset$ implies that $E \cap F = \emptyset$. The domain D is connected, so either E or F contains a limit point of the other. Assume E contains a point $\mathbf{x}$ with $\mathbf{x}$ a limit point of F. Since $\mathbf{x} \in E$, we have $f(\mathbf{x}) \in f(E) = f(f^{-1}(A)) = A$. Since $\mathbf{x}$ is a limit point of the set F and f is continuous, by Theorem 2.16 $f(\mathbf{x})$ is a limit point of the set $f(F) = f(f^{-1}(B)) = B$. Thus, A contains a point $f(\mathbf{x})$ which is a limit point of B. Therefore, R is connected. □

It follows that connectedness is a topological property.

▷ **Exercise 2.29.** Prove that X is connected if and only if X cannot be written as a union of two non-empty disjoint sets which are open relative to X.

▷ **Exercise 2.30.** Prove that X is connected if and only if X cannot be written as a union of two non-empty disjoint sets which are closed relative to X.

▷ **Exercise 2.31.** Give examples of sets A and B in $\mathbb{R}^2$ which satisfy:

(1) A and B are connected, but $A \cap B$ is not connected
(2) A and B are connected, but $A - B$ is not connected
(3) A and B are each not connected, but $A \cup B$ is connected.

2.6 Applications

In this section, we use the properties we have learned, continuity and connectedness, to derive some rather amusing and practical applications.

(2.30) Theorem. *If $f : [0,1] \longrightarrow [0,1]$ is a continuous function, then there is a point $t \in [0,1]$ with $f(t) = t$.*

Proof. Assume that $f(x)$ is never equal to x. Then consider the function $g : [0,1] \to \{\pm 1\}$ defined by

$$g(x) = \frac{f(x) - x}{|f(x) - x|}$$

Note that the quantity $\frac{a}{|a|}$ is equal to $+1$ if a is positive, and -1 if a is negative. Since $f(x) \neq x$, we have not divided by zero in the definition