of g, and the continuity of g follows from the continuity of f. Since we have assumed that $f(0) \neq 0$, it must be that $f(0) > 0$ and so $g(0) = +1$. Similarly, $f(1) \neq 1$, so $f(1) < 1$ and $g(1) = -1$. Thus, g is a continuous function taking the interval $[0, 1]$ onto the set $\{-1, 1\}$. By Theorem 2.28, the interval $[0, 1]$ is connected. Theorem 2.29 states that a continuous function must take a connected set to another connected set. However, g takes $[0, 1]$ to $\{-1, 1\}$, which is not connected. There is a contradiction unless there is a point $t \in [0, 1]$ with $f(t) = t$.

□

The theorem above is the one-dimensional version of a famous theorem, the Brouwer Fixed Point Theorem. This will be proven for higher dimensions in Chapter 10. Points such as t above with $f(t) = t$ are called *fixed points*, since f does not move t.

▷ **Exercise 2.32.** Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous function. Show that the *fixed point set* for f, $\mathcal{F}(f) = \{t \in \mathbb{R}^n : f(t) = t\}$, is closed.

(2.31) Definition. *A space X has the fixed point property if every continuous function $f : X \longrightarrow X$ has a fixed point.*

Theorem 2.30 above shows that the interval $[0, 1]$ has the fixed point property.

(2.32) Theorem. *The fixed point property is a topological property.*

▷ **Exercise 2.33.** Prove Theorem 2.32. [Hint: let X be a space with the fixed point property, and let Y be another space with a homeomorphism $f : Y \longrightarrow X$. Prove that if $g : Y \longrightarrow Y$ is any continuous function, then there is a point $y \in Y$ such that $g(y) = y$, by using the invertible function f to transfer the question to X.]

Let $\mathbb{S}^1$ denote the unit circle, i.e., $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

▷ **Exercise 2.34.** Give examples of functions which show that $\mathbb{R}$ and $\mathbb{S}^1$ do not have the fixed point property.

In higher dimensions, the n-dimensional sphere is defined as

$$\mathbb{S}^n = \{\mathbf{x} = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}.$$

(2.33) Definition. *Consider the n-sphere $\mathbb{S}^n$ as a subset of $\mathbb{R}^{n+1}$. The points $\mathbf{x} = (x_1, x_2, \dots, x_{n+1})$ and $-\mathbf{x} = (-x_1, -x_2, \dots, -x_{n+1})$ in $\mathbb{S}^n$ are called antipodal.*