1. Suppose that $\mathsf{C} \in \mathbb{C}^{n \times n}$ is invertible and the set $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subset \mathbb{C}^n$ is linearly independent. Prove that $\mathsf{C}S := \{\mathsf{C}\mathbf{w}_1, \ldots, \mathsf{C}\mathbf{w}_k\} \subset \mathbb{C}^n$ is linearly independent.

(d) Prove that $N(\mathsf{A}) = \text{span}\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$, where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the ordered columns of V.

(e) Prove that the non-zero singular values of A are equal to the square roots of the non-zero eigenvalues of $\mathsf{A}^H\mathsf{A}$ or $\mathsf{A}\mathsf{A}^H$. These last two matrices have the same non-zero eigenvalues.

(f) If $\mathsf{A} \in \mathbb{C}^{n\times n}_{\text{Her}}$, prove that the singular values are the absolute values of the eigenvalues of A.

6. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$. Prove that A is HPD if and only if $\mathsf{A} = \mathsf{B}^H\mathsf{B}$, where $\mathsf{B} \in \mathbb{C}^{n\times n}$ is invertible. (Hint for one of the directions, you may assume the appropriate factorization is valid.)

7. Let $\mu_{r,s} \in \mathbb{C}$, where $n \geq r > s \geq 1$, and define $\mathsf{L}^{(r,s)} = \left[\ell^{(r,s)}_{i,j}\right] \in \mathbb{C}^{n\times n}$ via

$$\ell^{(r,s)}_{i,j} = \begin{cases} 1 & \text{if} \quad i = j \\ \mu_{r,s} & \text{if} \quad i = r,\ j = s \\ 0 & \quad\text{otherwise} \end{cases} .$$

(a) Prove that $\mathsf{L}^{(r_1,s)}\mathsf{L}^{(r_2,s)} = \mathsf{L}^{(r_2,s)}\mathsf{L}^{(r_1,s)}$ for any $n \geq r_1 > r_2 > s \geq 1$.

(b) Prove or disprove (using a counter example) that $\mathsf{L}^{(r_1,s_1)}\mathsf{L}^{(r_2,s_2)} = \mathsf{L}^{(r_2,s_2)}\mathsf{L}^{(r_1,s_1)}$, for arbitrary indices that satisfy $n \geq r_\alpha > s_\alpha \geq 1$, $\alpha = 1, 2$.

(c) Define $\mathsf{L}^{(s)} := \mathsf{L}^{(n,s)} \cdots \mathsf{L}^{(s+1,s)}$, for each $1 \leq s < n$. Show that this product is invertible and find $\left(\mathsf{L}^{(s)}\right)^{-1}$.

(d) Write out the matrix $\mathsf{L} := \left(\mathsf{L}^{(1)}\right)^{-1} \cdots \left(\mathsf{L}^{(n-1)}\right)^{-1}$