Problem Set 3

This third homework set covers Fourier Series. The text which is covered is Chapter 3, Secs. 3.1 – 3.3. The problems from the text are:

3.2.2
3.3.2
3.3.5

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{L/2}^{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L}$$

$$= \frac{1}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) \tag{3.2.7}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx = \frac{-1}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/2}^{L}$$

$$= \frac{1}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right). \tag{3.2.8}$$

We omit simplifications that arise by noting that $\sin n\pi = 0$, $\cos n\pi = (-1)^n$, and so on.

EXERCISES 3.2

3.2.1. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$). Compare $f(x)$ to its Fourier series:

(a) $f(x) = 1$

∗(b) $f(x) = x^2$

(c) $f(x) = 1 + x$

∗(d) $f(x) = e^x$

(e) $f(x) = \begin{cases} x & x < 0 \\ 2x & x > 0 \end{cases}$

∗(f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 + x & x > 0 \end{cases}$

(g) $f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases}$

3.2.2. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$) and determine the Fourier coefficients:

∗(a) $f(x) = x$

(b) $f(x) = e^{-x}$

∗(c) $f(x) = \sin\dfrac{\pi x}{L}$

(d) $f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$

(e) $f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| > L/2 \end{cases}$

∗(f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$

(g) $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$

(a) $f(x) = 1$

(b) $f(x) = 1 + x$

(c) $f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases}$

*(d) $f(x) = e^x$

(e) $f(x) = \begin{cases} 2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$

3.3.2. For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients.

(a) $f(x) = \cos \pi x/L$ [Verify formula (3.3.13).]

(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

*(d) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

3.3.3. For the following functions, sketch the Fourier sine series of $f(x)$. Also, roughly sketch the sum of a *finite* number of nonzero terms (at least the first two) of the Fourier sine series:

(a) $f(x) = \cos \pi x/L$ [Use formula (3.3.13).]

(b) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = x$ [Use formula (3.3.12).]

3.3.4. Sketch the Fourier cosine series of $f(x) = \sin \pi x/L$. Briefly discuss.

3.3.5. For the following functions, sketch the Fourier cosine series of $f(x)$ and determine its Fourier coefficients:

(a) $f(x) = x^2$ (b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$ (c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

3.3.6. For the following functions, sketch the Fourier cosine series of $f(x)$. Also, roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier cosine series:

(a) $f(x) = x$ [Use formulas (3.3.22) and (3.3.23).]

(b) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ [Use carefully formulas (3.2.6) and (3.2.7).]

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ [*Hint*: Add the functions in parts (b) and (c).]