Prove that if D is the closed disc $|x| \leq 1$ in $\mathbb{R}^2$, then any map $f \in C^2[D \to D]$ has a fixed point: $f(x) = x$. The proof is by contradiction, and uses Stokes theorem. Follow the steps outlined below.

(1) Define a new map $F(x) = \frac{1}{r} f(\frac{x}{r})$ for $r > 1$ and $|x| \leq 1$. Show that F has no fixed points if r is small enough.

(2) Draw the ray from $F(x)$ to x (these are distict) and note where it cuts the circle $C : |x| = 1$. This point $G(x) = (\cos \phi, \sin \phi)$ depends smoothly on x, i.e. $\phi \in C^2(D)$; moreover, it reduces to the identity on C.

(3) Now compute

$$2(\text{Area}D) = 2\pi^2 = \int_C x_1 \, dx_2 - x_2 \, dx_1 = \int_C \nabla\phi \cdot dx$$

(4) Explain why the above is a contradiction?

Since this is an analysis problem, please be sure to be rigorous, and include as much detail as possible so that I can understand. Please also state if you are making use of some fact or theorem. Thanks!