EXERCISE 5-1

A

In Problems 1–10, construct a table of values for integer values of x over the indicated interval and then graph the function.

1. $y = 3^x; [-3, 3]$ **2.** $y = 5^x; [-2, 2]$

3. $y = (\frac{1}{3})^x = 3^{-x}; [-3, 3]$ **4.** $y = (\frac{1}{5})^x = 5^{-x}; [-2, 2]$

5. $g(x) = -3^{-x}; [-3, 3]$ **6.** $f(x) = -5^x; [-2, 2]$

7. $h(x) = 5(3^x); [-3, 3]$ **8.** $f(x) = 4(5^x); [-2, 2]$

9. $y = 3^{x+3} - 5; [-6, 0]$ **10.** $y = 5^{x+2} + 4; [-4, 0]$

In Problems 11–22, simplify.

11. $2^{5x-1}2^{3-2x}$ **12.** $5^{2x-3}5^{6-4x}$ **13.** $\frac{3^{2x-x}}{3^{3x-5x}}$

14. $\frac{7^{-3x-x}}{7^{-4x-6x}}$ **15.** $(4^x)^3$ **16.** $(x^3)^4$

17. $(10^2)^x(10^x)^2$ **18.** $2^x 2^x$ **19.** $\left(\frac{5^x}{4^y}\right)^3$

20. $\left(\frac{3^{2x}}{7^{3x}}\right)^2$ **21.** $\left(\frac{a^{-2}b^3c^{-1}}{a^{-3}b^2c^{-3}}\right)^2$ **22.** $\left(\frac{a^{-4}b^3c^2}{a^{-6}b^2c^{-1}}\right)^3$

B

In Problems 23–34, solve for x.

23. $3^{2x-5} = 3^{4x+2}$ **24.** $4^{4x+1} = 4^{2x-2}$

25. $10^{x^2+2} = 10^{2x+2}$ **26.** $5^{x^2-5} = 5^{3x+5}$

27. $(2x + 1)^3 = 8$ **28.** $(2x - 1)^5 = -32$

29. $5^{3x} = 25^{x+3}$ **30.** $4^{5x+1} = 16^{2x-1}$

31. $4^{2x+2} = 8^{x+2}$ **32.** $100^{2x+4} = 1{,}000^{x+4}$

33. $100^{x^2} = 10^{5x-3}$ **34.** $3^{x^2} = 9^{x+4}$

35. Find all real numbers a such that $a^2 = a^{-2}$. Explain why this does not violate the second exponential function property in the box on page 358.

36. Find real numbers a and b such that $a \neq b$ but $a^4 = b^4$. Explain why this does not violate the third exponential function property in the box on page 358.

Graph each function in Problems 37–46 by constructing a table of values.

* *Check Problems 37–46 with a graphing utility.*

37. $G(t) = 3^{t/100}$ **38.** $f(t) = 2^{t/10}$ **39.** $y = 11(3^{-x/2})$

40. $y = 7(2^{-2x})$ **41.** $g(x) = 2^{-|x|}$ **42.** $f(x) = 2^{|x|}$

43. $y = 1{,}000(1.08)^x$ **44.** $y = 100(1.03)^x$

45. $y = 2^{-x^2}$ **46.** $y = 3^{-x^2}$

*Please note that use of graphing utility is not required to complete these exercises. Checking them with a g.u. is optional.

C

In Problems 47–50, simplify.

47. $(6^x + 6^{-x})(6^x - 6^{-x})$ **48.** $(3^x - 3^{-x})(3^x + 3^{-x})$

49. $(6^x + 6^{-x})^2 - (6^x - 6^{-x})^2$ **50.** $(3^x - 3^{-x})^2 + (3^x + 3^{-x})^2$

Graph each function in Problems 51–54 by constructing a table of values.

Check Problems 51–54 with a graphing utility.

51. $m(x) = x(3^{-x})$ **52.** $h(x) = x(2^x)$

53. $f(x) = \frac{2^x + 2^{-x}}{2}$ **54.** $g(x) = \frac{3^x + 3^{-x}}{2}$

In Problems 55–58:

(A) Approximate the real zeros of each function to two decimal places.

(B) Investigate the behavior of each function as $x \to \infty$ and $x \to -\infty$ and find any horizontal asymptotes.

55. $f(x) = 3^x - 5$ **56.** $f(x) = 4 + 2^{-x}$

57. $f(x) = 1 + x + 10^x$ **58.** $f(x) = 8 - x^2 + 2^{-x}$

APPLICATIONS

59. Gaming. A person bets on red and black on a roulette wheel using a *Martingale strategy*. That is, a \$2 bet is placed on red, and the bet is doubled each time until a win occurs. The process is then repeated. If black occurs n times in a row, then $L = 2^n$ dollars is lost on the nth bet. Graph this function for $1 \le n \le 10$. Even though the function is defined only for positive integers, points on this type of graph are usually jointed with a smooth curve as a visual aid.

60. Bacterial Growth. If bacteria in a certain culture double every $\frac{1}{2}$ hour, write an equation that gives the number of bacteria N in the culture after t hours, assuming the culture has 100 bacteria at the start. Graph the equation for $0 \le t \le 5$.

61. Population Growth. Because of its short life span and frequent breeding, the fruit fly *Drosophila* is used in some genetic studies. Raymond Pearl of Johns Hopkins University,

for example, studied 300 successive generations of descendants of a single pair of *Drosophila* flies. In a laboratory situation with ample food supply and space, the doubling time for a particular population is 2.4 days. If we start with 5 male and 5 female flies, how many flies should we expect to have in:
(A) 1 week? (B) 2 weeks?

62. Population Growth. If Kenya has a population of about 30,000,000 people and a doubling time of 19 years and if the growth continues at the same rate, find the population in:
(A) 10 years (B) 30 years
Compute answers to 2 significant digits.

63. Insecticides. The use of the insecticide DDT is no longer allowed in many countries because of its long-term adverse effects. If a farmer uses 25 pounds of active DDT, assuming its half-life is 12 years, how much will still be active after:
(A) 5 years? (B) 20 years?
Compute answers to 2 significant digits.

64. Radioactive Tracers. The radioactive isotope technetium 99m (^{99m}Tc) is used in imaging the brain. The isotope has a half-life of 6 hours. If 12 milligrams are used, how much will be present after:
(A) 3 hours? (B) 24 hours?
Compute answers to 3 significant digits.

65. Finance. Suppose $4,000 is invested at 11% compounded weekly. How much money will be in the account in:
(A) $\frac{1}{2}$ year? (B) 10 years?
Compute answers to the nearest cent.

66. Finance. Suppose $2,500 is invested at 7% compounded quarterly. How much money will be in the account in:
(A) $\frac{3}{4}$ year? (B) 15 years?
Compute answers to the nearest cent.

★ **67. Finance.** A couple just had a new child. How much should they invest now at 8.25% compounded daily in order to have $40,000 for the child's education 17 years from now? Compute the answer to the nearest dollar.

★ **68. Finance.** A person wishes to have $15,000 cash for a new car 5 years from now. How much should be placed in an account now if the account pays 9.75% compounded weekly? Compute the answer to the nearest dollar.

★ **69. Finance.** Will an investment of $10,000 at 8.9% compounded daily ever be worth more at the end of a quarter than an investment of $10,000 at 9% compounded quarterly? Explain.

★ **70. Finance.** A sum of $5,000 is invested at 13% compounded semiannually. Suppose that a second investment of $5,000 is made at interest rate r compounded daily. For which values of r, to the nearest tenth of a percent, is the second investment better than the first? Discuss.

SECTION 5-2 The Exponential Function with Base *e*

- Base *e* Exponential Function
- Growth and Decay Applications Revisited
- Continuous Compound Interest
- A Comparison of Exponential Growth Phenomena

Until now the number π has probably been the most important irrational number you have encountered. In this section we will introduce another irrational number, e, that is just as important in mathematics and its applications.

• Base *e* Exponential Function

The following expression is important to the study of calculus and, as we will see later in this section, also is closely related to the compound interest formula discussed in the preceding section:

$$\left(1 + \frac{1}{m}\right)^m$$

> **Continuous Compound Interest Formula**
>
> If a principal P is invested at an annual rate r compounded continuously, then the amount A in the account at the end of t years is given by
>
> $$A = Pe^{rt}$$
>
> The annual rate r is expressed as a decimal.

EXAMPLE 4 **Continuous Compound Interest**

If \$100 is invested at an annual rate of 8% compounded continuously, what amount, to the nearest cent, will be in the account after 2 years?

Solution Use the continuous compound interest formula to find A when $P = \$100$, $r = 0.08$, and $t = 2$:

$$\begin{aligned} A &= Pe^{rt} \\ &= \$100e^{(0.08)(2)} \qquad \text{8\% is equivalent to } r = 0.08. \\ &= \$117.35 \end{aligned}$$

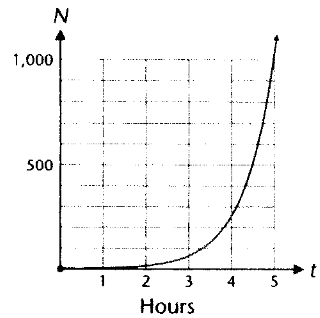
Compare this result with the values calculated in Table 2.

Matched Problem 4 What amount will an account have after 5 years if \$100 is invested at an annual rate of 12% compounded annually? Quarterly? Continuously? Compute answers to the nearest cent.

The continuous compound interest formula may also be used to model short-term population growth. If a population P is assumed to grow continuously at an annual rate r, then the population A at the end of t years is given by $A = Pe^{rt}$.

• A Comparison of Exponential Growth Phenomena

The equations and graphs given in Table 3 compare several widely used growth models. These are divided basically into two groups: unlimited growth and limited growth. Following each equation and graph is a short, incomplete list of areas in which the models are used. We have only touched on a subject that has been extensively developed and which you are likely to study in greater depth in the future.

2. 

3. 119.4 mg
4. Annually: \$176.23; quarterly: \$180.61; continuously: \$182.21

EXERCISE 5-2

A

In Problems 1–6, construct a table of values for integer values of x over the indicated interval and then graph the function.

Check Problems 1–6 with a graphing utility.

1. $y = -e^x$; $[-3, 3]$ **2.** $y = -e^{-x}$; $[-3, 3]$

3. $y = 10e^{0.2x}$; $[-5, 5]$ **4.** $y = 100e^{0.1x}$; $[-5, 5]$

5. $f(t) = 100e^{-0.1t}$; $[-5, 5]$ **6.** $g(t) = 10e^{-0.2t}$; $[-5, 5]$

In Problems 7–12, simplify.

7. $e^{2x}e^{-3x}$ **8.** $(e^{-x})^4$ **9.** $(e^x)^3$

10. $e^{-4x}e^{6x}$ **11.** $\dfrac{e^{5x}}{e^{2x+1}}$ **12.** $\dfrac{e^{4-3x}}{e^{2-5x}}$

13. (A) Explain what is wrong with the following reasoning about the expression $[1 + (1/m)]^m$: As m gets large, $1 + (1/m)$ approaches 1 because $1/m$ approaches 0, and 1 raised to any power is 1, so $[1 + (1/m)]^m$ approaches 1.
(B) Which number does $[1 + (1/m)]^m$ approach as $m \to \infty$?

14. (A) Explain what is wrong with the following reasoning about the expression $[1 + (1/m)]^m$: If $b > 1$, then the exponential function $b^x \to \infty$ as $x \to \infty$, and $1 + (1/m)$ is greater than 1, so $[1 + (1/m)]^m$ approaches infinity as $m \to \infty$.
(B) Which number does $[1 + (1/m)]^m$ approach as $m \to \infty$?

B

In Problems 15–22, graph each function by constructing a table of values.

Check Problems 15–22 with a graphing utility.

15. $y = 2 + e^{x-2}$ **16.** $y = -3 + e^{1-x}$

17. $y = e^{-|x|}$ **18.** $y = e^{|x|}$

19. $M(x) = e^{x/2} + e^{-x/2}$ **20.** $C(x) = \dfrac{e^x + e^{-x}}{2}$

21. $N = \dfrac{200}{1 + 3e^{-t}}$ **22.** $N = \dfrac{100}{1 + e^{-t}}$

In Problems 23–28, simplify.

23. $\dfrac{-2x^3e^{-2x} - 3x^2e^{-2x}}{x^6}$ **24.** $\dfrac{5x^4e^{5x} - 4x^3e^{5x}}{x^8}$

25. $(e^x + e^{-x})^2 + (e^x - e^{-x})^2$

26. $e^x(e^{-x} + 1) - e^{-x}(e^x + 1)$

27. $\dfrac{e^{-x}(e^x - e^{-x}) + e^{-x}(e^x + e^{-x})}{e^{-2x}}$

28. $\dfrac{e^x(e^x + e^{-x}) - (e^x - e^{-x})e^x}{e^{2x}}$

In Problems 29–32, solve each equation. [Remember: $e^{-x} \neq 0$.]

29. $2xe^{-x} = 0$ **30.** $(x - 3)e^x = 0$

31. $x^2e^x - 5xe^x = 0$ **32.** $3xe^{-x} + x^2e^{-x} = 0$

C

One of the most important functions in statistics is the ***normal probability density function***

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

where μ is the ***mean*** *and σ is the* ***standard deviation.*** *The graph of this function is the "bell-shaped" curve that instructors refer to when they say that they are grading on a curve.*

Graph the related functions given in Problems 33 and 34.

33. $f(x) = e^{-x^2}$ **34.** $g(x) = \dfrac{1}{\sqrt{\pi}}e^{-x^2/2}$

35. Given $f(s) = (1 + s)^{1/s}$, $s \neq 0$:
(A) Complete the tables below to four decimal places.
(B) What does $(1 + s)^{1/s}$ seem to tend to as s approaches 0?

s	$f(s)$	s	$f(s)$
−0.5	4.0000	0.5	2.2500
−0.2	3.0518	0.2	2.4883
−0.1		0.1	
−0.01		0.01	
−0.001		0.001	
−0.0001		0.0001	

36. Refer to Problem 35. Graph $f(s) = (1 + s)^{1/s}$ for s in $[-0.5, 0) \cup (0, 0.5]$.

Problems 37–40 require the use of a graphing utility.

It is common practice in many applications of mathematics to approximate nonpolynomial functions with appropriately selected polynomials. For example, the polynomials in Problems 37–40, called ***Taylor polynomials,*** *can be used to approximate the exponential function $f(x) = e^x$. To illustrate this approximation graphically, in each problem graph $f(x) = e^x$ and the indicated polynomial in the same viewing window, $-4 \le x \le 4$ and $-5 \le y \le 50$.*

37. $P_1(x) = 1 + x + \frac{1}{2}x^2$

38. $P_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

39. $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

40. $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$

41. Investigate the behavior of the functions $f_1(x) = x/e^x$, $f_2(x) = x^2/e^x$, and $f_3(x) = x^3/e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $f_n(x) = x^n/e^x$, where n is any positive integer.

42. Investigate the behavior of the functions $g_1(x) = xe^x$, $g_2(x) = x^2e^x$, and $g_3(x) = x^3e^x$ as $x \to \infty$ and as $x \to -\infty$, and find any horizontal asymptotes. Generalize to functions of the form $g_n(x) = x^ne^x$, where n is any positive integer.

APPLICATIONS

43. **Population Growth.** If the world population is about 6 billion people now and if the population grows continuously at an annual rate of 1.7%, what will the population be in 10 years? Compute the answer to 2 significant digits.

44. **Population Growth.** If the population in Mexico is around 100 million people now and if the population grows continuously at an annual rate of 2.3%, what will the population be in 8 years? Compute the answer to 2 significant digits.

45. **Population Growth.** In 1996 the population of Russia was 148 million and the population of Nigeria was 104 million. If the populations of Russia and Nigeria grow continuously at annual rates of −0.62% and 3.0%, respectively, when will Nigeria have a greater population than Russia?

46. **Population Growth.** In 1996 the population of Germany was 84 million and the population of Egypt was 64 million. If the populations of Germany and Egypt grow continuously at annual rates of −0.15% and 1.9%, respectively, when will Egypt have a greater population than Germany?

47. **Space Science.** Radioactive isotopes, as well as solar cells, are used to supply power to space vehicles. The isotopes gradually lose power because of radioactive decay. On a particular space vehicle the nuclear energy source has a power output of P watts after t days of use as given by

$$P = 75e^{0.0035t}$$

Graph this function for $0 \le t \le 100$.

48. **Earth Science.** The atmospheric pressure P, in pounds per square inch, decreases exponentially with altitude h, in miles above sea level, as given by

$$P = 14.7e^{0.21h}$$

Graph this function for $0 \le h \le 10$.

49. **Marine Biology.** Marine life is dependent upon the microscopic plant life that exists in the *photic zone*, a zone that goes to a depth where about 1% of the surface light still remains. Light intensity I relative to depth d, in feet, for one of the clearest bodies of water in the world, the Sargasso Sea in the West Indies, can be approximated by

$$I = I_0e^{0.00942d}$$

where I_0 is the intensity of light at the surface. What percentage of the surface light will reach a depth of:
(A) 50 feet? (B) 100 feet?

50. **Marine Biology.** Refer to Problem 49. In some waters with a great deal of sediment, the photic zone may go down only 15 to 20 feet. In some murky harbors, the intensity of light d feet below the surface is given approximately by

$$I = I_0e^{-0.23d}$$

What percentage of the surface light will reach a depth of:
(A) 10 feet? (B) 20 feet?

51. **Money Growth.** If you invest $5,250 in an account paying 11.38% compounded continuously, how much money will be in the account at the end of:
(A) 6.25 years? (B) 17 years?

52. **Money Growth.** If you invest $7,500 in an account paying

8.35% compounded continuously, how much money will be in the account at the end of:
(A) 5.5 years? (B) 12 years?

53. Money Growth. *Barron's*, a national business and financial weekly, published the following "Top Savings Deposit Yields" for $2\frac{1}{2}$-year certificate of deposit accounts:

Gill Saving	8.30% (CC)
Richardson Savings and Loan	8.40% (CQ)
USA Savings	8.25% (CD)

where CC represents compounded continuously, CQ compounded quarterly, and CD compounded daily. Compute the value of $1,000 invested in each account at the end of $2\frac{1}{2}$ years.

54. Money Growth. Refer to Problem 53. In another issue of *Barron's*, 1-year certificate of deposit accounts included:

Alamo Savings	8.25% (CQ)
Lamar Savings	8.05% (CC)

Compute the value of $10,000 invested in each account at the end of 1 year.

★ **55. Present Value.** A promissory note will pay $30,000 at maturity 10 years from now. How much should you be willing to pay for the note now if the note gains value at a rate of 9% compounded continuously?

★ **56. Present Value.** A promissory note will pay $50,000 at maturity $5\frac{1}{2}$ years from now. How much should you be willing to pay for the note now if the note gains value at a rate of 10% compounded continuously?

57. AIDS Epidemic. In June of 1996 the World Health Organization estimated that 7.7 million cases of AIDS (acquired immunodeficiency syndrome) had occurred worldwide since the beginning of the epidemic. Assuming that the disease spreads continuously at an annual rate of 17%, estimate the total number of AIDS cases which will have occurred by June of the year:
(A) 2000 (B) 2004

58. AIDS Epidemic. In June 1996 the World Health Organization estimated that 28 million people worldwide had been infected with HIV (human immunodeficiency virus) since the beginning of the AIDS epidemic. Assuming that HIV infection spreads continuously at an annual rate of 19%, estimate the total number of people who will have been infected with HIV by June of the year:
(A) 2000 (B) 2004

★ **59. Learning Curve.** People assigned to assemble circuit boards for a computer manufacturing company undergo on-the-job training. From past experience it was found that the learning curve for the average employee is given by

$$N = 40(1 - e^{-0.12t})$$

where N is the number of boards assembled per day after t days of training. Graph this function for $0 \le t \le 30$. What is the maximum number of boards an average employee can be expected to produce in 1 day?

★ **60. Advertising.** A company is trying to expose a new product to as many people as possible through television advertising in a large metropolitan area with 2 million possible viewers. A model for the number of people N, in millions, who are aware of the product after t days of advertising was found to be

$$N = 2(1 - e^{-0.037t})$$

Graph this function for $0 \le t \le 50$. What value does N tend to as t increases without bound?

61. Newton's Law of Cooling. This law states that the rate at which an object cools is proportional to the difference in temperature between the object and its surrounding medium. The temperature T of the object t hours later is given by

$$T = T_m + (T_0 - T_m)e^{-kt}$$

where T_m is the temperature of the surrounding medium and T_0 is the temperature of the object at $t = 0$. Suppose a bottle of wine at a room temperature of 72°F is placed in the refrigerator to cool before a dinner party. If the temperature in the refrigerator is kept at 40°F and $k = 0.4$, find the temperature of the wine, to the nearest degree, after 3 hours. (In Exercise 5-5 we will find out how to determine k.)

62. Newton's Law of Cooling. Refer to Problem 61. What is the temperature, to the nearest degree, of the wine after 5 hours in the refrigerator?

★ **63. Photography.** An electronic flash unit for a camera is activated when a capacitor is discharged through a filament of wire. After the flash is triggered, and the capacitor is discharged, the circuit (see the figure) is connected and the battery pack generates a current to recharge the capacitor. The time it takes for the capacitor to recharge is called the *recycle time*. For a particular flash unit using a 12-volt battery pack, the charge q, in coulombs, on the capacitor t seconds after recharging has started is given by

$$q = 0.0009(1 - e^{-0.2t})$$

Graph this function for $0 \le t \le 10$. Estimate the maximum charge on the capacitor.