EXERCISE 3-4

A

Problems 1–6 refer to functions f, g, h, k, p, and q given by the following graphs. (Assume the graphs continue as indicated beyond the parts shown.)

1. For the function f, find:
(A) Domain (B) Range
(C) x intercepts (D) y intercept
(E) Intervals over which f is increasing
(F) Intervals over which f is decreasing
(G) Intervals over which f is constant
(H) Any points of discontinuity

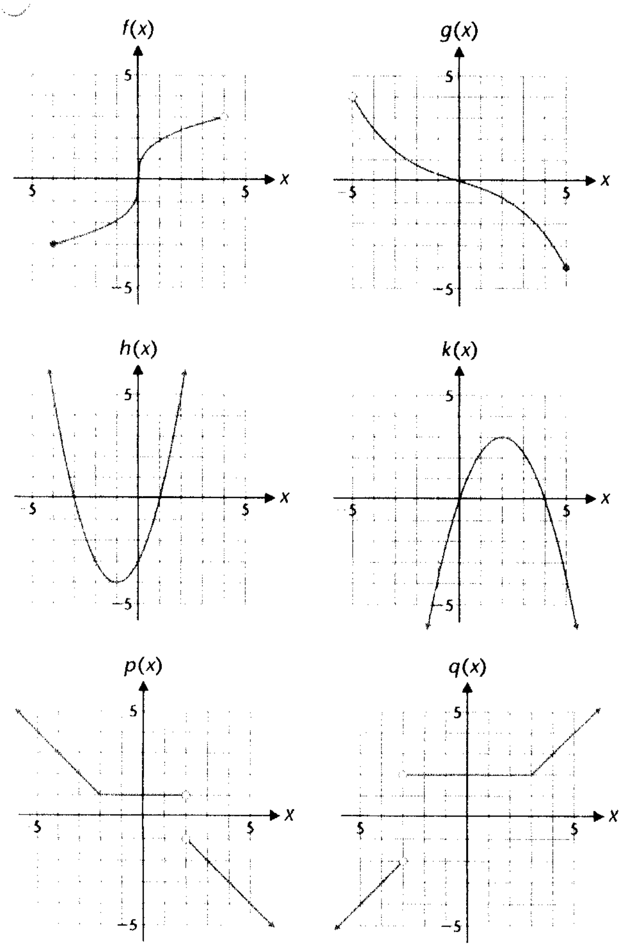
2. Repeat Problem 1 for the function g.

3. Repeat Problem 1 for the function h.

4. Repeat Problem 1 for the function k.

5. Repeat Problem 1 for the function p.

6. Repeat Problem 1 for the function q.



Problems 7–12 describe the graph of a continuous function f over the interval [−5, 5]. Sketch the graph of a function that is consistent with the given information.

7. The function f is increasing on $[-5, -2]$, constant on $[-2, 2]$, and decreasing on $[2, 5]$.

8. The function f is decreasing on $[-5, -2]$, constant on $[-2, 2]$, and increasing on $[2, 5]$.

9. The function f is decreasing on $[-5, -2]$, constant on $[-2, 2]$, and decreasing on $[2, 5]$.

10. The function f is increasing on $[-5, -2]$, constant on $[-2, 2]$, and increasing on $[2, 5]$.

11. The function f is decreasing on $[-5, -2]$, increasing on $[-2, 2]$, and decreasing on $[2, 5]$.

12. The function f is increasing on $[-5, -2]$, decreasing on $[-2, 2]$, and increasing on $[2, 5]$.

In Problems 13–16, find the slope and intercepts, and then sketch the graph.

13. $f(x) = 2x + 4$ **14.** $f(x) = 3x - 3$

15. $f(x) = -\frac{1}{2}x - \frac{5}{3}$ **16.** $f(x) = -\frac{3}{4}x + \frac{6}{5}$

In Problems 17 and 18, find a linear function f satisfying the given conditions.

17. $f(-2) = 7$ and $f(4) = -2$

18. $f(-3) = -2$ and $f(5) = 4$

B

In Problems 19–22, graph, finding the axis, vertex, maximum or minimum, and range.

19. $f(x) = (x - 3)^2 + 2$

20. $f(x) = \frac{1}{2}(x + 2)^2 - 4$

21. $f(x) = -(x + 3)^2 - 2$

22. $f(x) = -(x - 2)^2 + 4$

In Problems 23–26, graph, finding the axis, vertex, x intercepts, and y intercept.

23. $f(x) = x^2 - 4x - 5$ **24.** $f(x) = x^2 - 6x + 5$

25. $f(x) = -x^2 + 6x$ **26.** $f(x) = -x^2 + 2x + 8$

In Problems 27–30, graph, finding the axis, vertex, intervals over which f is increasing, and intervals over which f is decreasing.

27. $f(x) = x^2 + 6x + 11$ **28.** $f(x) = x^2 - 8x + 14$

29. $f(x) = -x^2 + 6x - 6$ **30.** $f(x) = -x^2 - 10x - 24$

In Problems 31–38, graph, finding the domain, range, and any points of discontinuity.

31. $f(x) = \begin{cases} x+1 & \text{if } -1 \le x < 0 \\ -x+1 & \text{if } 0 \le x \le 1 \end{cases}$

32. $f(x) = \begin{cases} x & \text{if } -2 \le x < 1 \\ -x+2 & \text{if } 1 \le x \le 2 \end{cases}$

33. $f(x) = \begin{cases} -2 & \text{if } -3 \le x < -1 \\ 4 & \text{if } -1 < x \le 2 \end{cases}$

34. $f(x) = \begin{cases} 1 & \text{if } -2 \le x < 2 \\ -3 & \text{if } 2 < x \le 5 \end{cases}$

35. $f(x) = \begin{cases} x+2 & \text{if } x < -1 \\ x-2 & \text{if } x \ge -1 \end{cases}$

36. $f(x) = \begin{cases} -1-x & \text{if } x \le 2 \\ 5-x & \text{if } x > 2 \end{cases}$

37. $g(x) = \begin{cases} x^2+1 & \text{if } x < 0 \\ -x^2-1 & \text{if } x > 0 \end{cases}$

38. $h(x) = \begin{cases} -x^2-2 & \text{if } x < 0 \\ x^2+2 & \text{if } x > 0 \end{cases}$

C

In Problems 39–44, graph, finding the axis, vertex, maximum or minimum of $f(x)$, range, intercepts, intervals over which f is increasing, and intervals over which f is decreasing.

39. $f(x) = \frac{1}{2}x^2 + 2x + 3$

40. $f(x) = 2x^2 - 12x + 14$

41. $f(x) = 4x^2 - 12x + 9$

42. $f(x) = -\frac{1}{2}x^2 + 4x - 10$

43. $f(x) = -2x^2 - 8x - 2$

44. $f(x) = -4x^2 - 4x - 1$

In Problems 45–50, find a piecewise definition of f that does not involve the absolute value function (see Example 5). Sketch the graph, and find the domain, range, and any points of discontinuity.

Check your graphs in Problems 45–50 by graphing the given definition of f on a graphing utility.

45. $f(x) = \dfrac{|x|}{x}$

46. $f(x) = x|x|$

47. $f(x) = x + \dfrac{|x-1|}{x-1}$

48. $f(x) = x + 2\dfrac{|x+1|}{x+1}$

49. $f(x) = |x| + |x-2|$

50. $f(x) = |x| - |x-3|$

In Problems 51–56, write a piecewise definition for f (see the discussion of Fig. 14 in this section) and sketch the graph of f. Include sufficient intervals to clearly illustrate both the definition and the graph. Find the domain, range, and any points of discontinuity.

Check your graphs in Problems 51–56 by graphing the given definition of f on a graphing utility.

51. $f(x) = [\![x/2]\!]$

52. $f(x) = [\![x/3]\!]$

53. $f(x) = [\![3x]\!]$

54. $f(x) = [\![2x]\!]$

55. $f(x) = x - [\![x]\!]$

56. $f(x) = [\![x]\!] - x$

57. Given that f is a quadratic function with $\min f(x) = f(2) = 4$, find the axis, vertex, range, and x intercepts.

58. Given that f is a quadratic function with $\max f(x) = f(-3) = -5$, find the axis, vertex, range, and x intercepts.

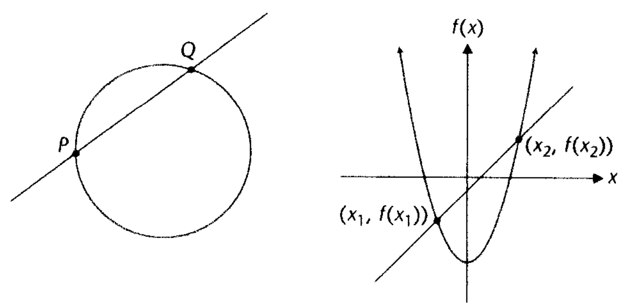
59. The function f is continuous and increasing on the interval $[1, 9]$ with $f(1) = -5$ and $f(9) = 4$.
 (A) Sketch a graph of f that is consistent with the given information.
 (B) How many times does your graph cross the x axis? Could the graph cross more times? Fewer times? Support your conclusions with additional sketches and/or verbal arguments.

60. Repeat Problem 59 if the function does not have to be continuous.

61. The function f is continuous on the interval $[-5, 5]$ with $f(-5) = -4$, $f(1) = 3$, and $f(5) = -2$.
 (A) Sketch a graph of f that is consistent with the given information.
 (B) How many times does your graph cross the x axis? Could the graph cross more times? Fewer times? Support your conclusions with additional sketches and/or verbal arguments.

62. Repeat Problem 61 if f is continuous on $[-8, 8]$ with $f(-8) = -6$, $f(-4) = 3$, $f(3) = -2$, and $f(8) = 5$.

Problems 63–66 are calculus-related. In geometry, a line that intersects a circle in two distinct points is called a secant line, as shown in figure (a). In calculus, the line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is called a ***secant line*** *for the graph of the function f, as shown in figure (b).*



Secant line for a circle (a)

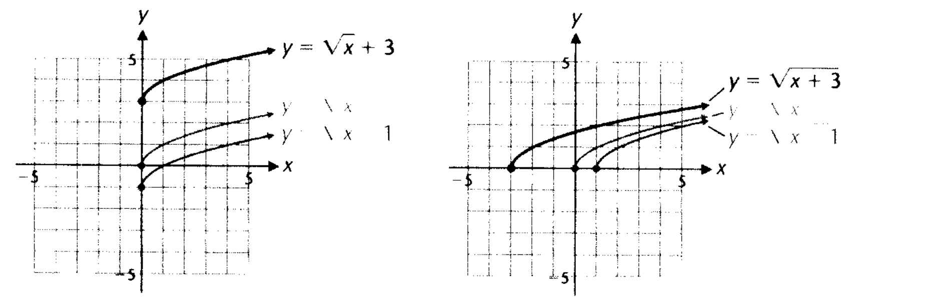
Secant line for the graph of a function (b)

In Problems 63 and 64, find the equation of the secant line through the indicated points on the graph of f. Graph f and the secant line on the same coordinate system.

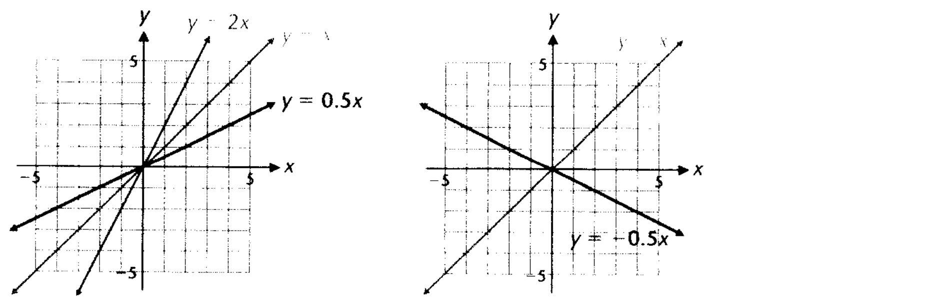
63. $f(x) = x^2 - 4$; $(-1, -3)$, $(3, 5)$

Answers to Matched Problems

1. $(f + g)(x) = \sqrt{x} + \sqrt{10 - x}$, $(f - g)(x) = \sqrt{x} - \sqrt{10 - x}$, $(fg)(x) = \sqrt{10x - x^2}$, $(f/g)(x) = \sqrt{x/(10 - x)}$; the functions $f + g$, $f - g$, and fg have domain $[0, 10]$, the domain of f/g is $[0, 10)$
2. $(f \circ g)(x) = x$, domain $= (-\infty, \infty)$
   $(g \circ f)(x) = x$, domain $= (-\infty, \infty)$
3. $(f \circ g)(x) = \sqrt{10 - x}$; domain: $x \geq 1$ and $x \leq 10$ or $[1, 10]$
4. $h(x) = (f \circ g)(x)$, where $f(x) = \sqrt{x}$ and $g(x) = 4x - 7$
5. (A) The graph of $y = \sqrt{x} + 3$ is the same as the graph of $y = \sqrt{x}$ shifted upward 3 units, and the graph of $y = \sqrt{x} - 1$ is the same as the graph of $y = \sqrt{x}$ shifted downward 1 unit. The figure confirms these conclusions.
   (B) The graph of $y = \sqrt{x + 3}$ is the same as the graph of $y = \sqrt{x}$ shifted to the left 3 units, and the graph of $y = \sqrt{x - 1}$ is the same as the graph of $y = \sqrt{x}$ shifted to the right 1 unit. The figure confirms these conclusions.



6. $G(x) = (x + 3)^3$, $H(x) = (x - 1)^3$, $M(x) = x^3 + 3$, $N(x) = x^3 - 4$
7. (A) The graph of $y = 2x$ is a vertical expansion of the graph of $y = x$, and the graph of $y = 0.5x$ is a vertical contraction of the graph of $y = x$. The figure confirms these conclusions.
   (B) The graph of $y = -0.5x$ is a vertical contraction and a reflection in the x axis of the graph of $y = x$. The figure confirms this conclusion.



8. The graph of function h is a reflection in the x axis and a horizontal translation of 3 units to the left of the graph of $y = x^3$. An equation for h is $h(x) = -(x + 3)^3$.

EXERCISE 3-5

A

Without looking back in the text, indicate the domain and range of each of the following functions. (Making rough sketches on scratch paper may help.)

1. $h(x) = -\sqrt{x}$ **2.** $m(x) = -\sqrt[3]{x}$ **3.** $g(x) = -2x^2$

4. $f(x) = -0.5|x|$ **5.** $F(x) = -0.5x^3$ **6.** $G(x) = 4x^3$

In Problems 7–10, for the indicated functions f and g, find the functions f + g, f − g, fg, and f/g, and find their domains.

7. $f(x) = 4x$; $g(x) = x + 1$

8. $f(x) = 3x$; $g(x) = x - 2$

9. $f(x) = 2x^2$; $g(x) = x^2 + 1$

10. $f(x) = 3x; \quad g(x) = x^2 + 4$

In Problems 11–14, for the indicated functions f and g, find the functions $f \circ g$ and $g \circ f$, and find their domains.

11. $f(x) = x^2 + 3; \quad g(x) = x^2 - 4x$

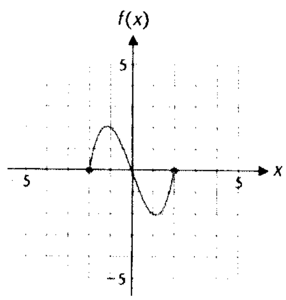
12. $f(x) = x^2 - 5x; \quad g(x) = x^2 + 1$

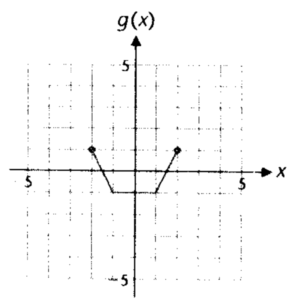
13. $f(x) = 2x^{2/3}; \quad g(x) = x^3 - 1$

14. $f(x) = 4 - x^3; \quad g(x) = 3x^{1/3}$

Problems 15–22 refer to the functions f and g given by the graphs below (the domain of each function is $[-2, 2]$).

Use the graph of f or g, as required, to graph each given function.





15. $f(x) + 2$	**16.** $g(x) - 1$	**17.** $g(x + 2)$
18. $f(x - 1)$	**19.** $-f(x)$	**20.** $-g(x)$
21. $2g(x)$	**22.** $\frac{1}{2}f(x)$	

B

In Problems 23–28, indicate how the graph of each function is related to the graph of one of the six basic functions in Figure 2. Sketch a graph of each function.

Check your descriptions and graphs in Problems 23–28 by graphing each function on a graphing utility.

23. $g(x) = -|x + 2|$ **24.** $h(x) = -|x - 4|$

25. $f(x) = (x - 2)^2 - 4$ **26.** $m(x) = (x + 1)^2 + 3$

27. $f(x) = 4 - 2\sqrt{x}$ **28.** $g(x) = -2 + 3\sqrt[3]{x}$

In Problems 29–34, for the indicated functions f and g, find the functions $f + g$, $f - g$, fg, and f/g, and find their domains.

29. $f(x) = \sqrt{x + 2}; \quad g(x) = \sqrt{4 - x}$

30. $f(x) = \sqrt{5 - x}; \quad g(x) = \sqrt{x + 1}$

31. $f(x) = 5 - 2\sqrt{x}; \quad g(x) = 3 - \sqrt{x}$

32. $f(x) = 3\sqrt{x} + 6; \quad g(x) = \sqrt{x} - 1$

33. $f(x) = \sqrt{x^2 + x - 2}; \quad g(x) = \sqrt{24 + 2x - x^2}$

34. $f(x) = \sqrt{x^2 + 3x - 10}; \quad g(x) = \sqrt{x^2 - x - 12}$

In Problems 35–40, for the indicated functions f and g, find the functions $f \circ g$ and $g \circ f$, and find their domains.

35. $f(x) = x + 2; \quad g(x) = \sqrt{4 - x}$

36. $f(x) = \sqrt{x + 1}; \quad g(x) = x - 2$

37. $f(x) = x + 3; \quad g(x) = \dfrac{1}{x - 2}$

38. $f(x) = \dfrac{x}{x + 4}; \quad g(x) = 2 - x$

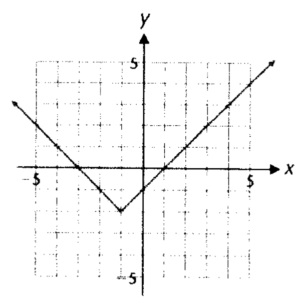
39. $f(x) = |x + 2|; \quad g(x) = \dfrac{x}{x - 3}$

40. $f(x) = \dfrac{x}{x - 4}; \quad g(x) = |x + 3|$

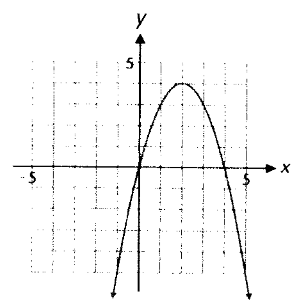
Each graph in Problems 41–46 is the result of applying a sequence of transformations to the graph of one of the six basic functions in Figure 2. Identify the basic function and describe the transformation verbally. Write an equation for the given graph.

Check your equations in Problems 41–46 by graphing each on a graphing utility.

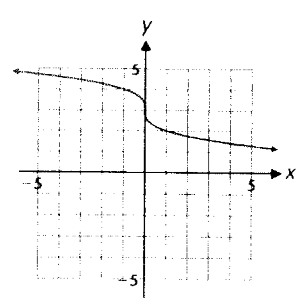
41.



42.



43.



$f(a) = f(b)$	Assume second components are equal.
$2a - 1 = 2b - 1$	Evaluate $f(a)$ and $f(b)$.
$2a = 2b$	Simplify.
$a = b$	Conclusion: f is one-to-one

Thus, by Definition 1, f is a one-to-one function.

Matched Problem 1 Determine whether f is a one-to-one function for:

(A) $f(x) = 4 - x^2$ (B) $f(x) = 4 - 2x$

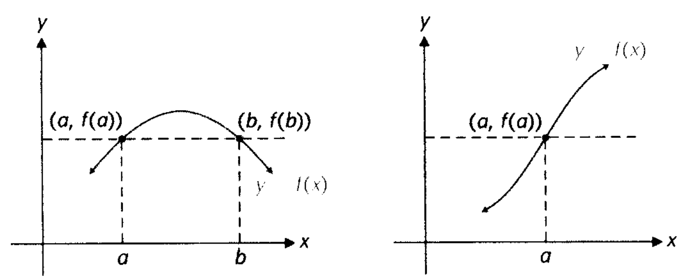
The methods used in the solution of Example 1 can be stated as a theorem.

Theorem 1 **One-to-One Functions**

1. If $f(a) = f(b)$ for at least one pair of domain values a and b, $a \neq b$, then f is not one-to-one.
2. If the assumption $f(a) = f(b)$ always implies that the domain values a and b are equal, then f is one-to-one.

Applying Theorem 1 is not always easy—try testing $f(x) = x^3 + 2x + 3$, for example. However, if we are given the graph of a function, then there is a simple graphic procedure for determining if the function is one-to-one. If a horizontal line intersects the graph of a function in more than one point, then the function is not one-to-one, as shown in Figure 1(a). However, if each horizontal line intersects the graph in one point, or not at all, then the function is one-to-one, as shown in Figure 1(b). These observations form the basis for the *horizontal line test*.

FIGURE 1 Intersections of graphs and horizontal lines.



$f(a) = f(b)$ for $a \neq b$
f is not one-to-one
(a)

Only one point has ordinate
$f(a)$; f is one-to-one
(b)

EXERCISE 3-6

A

Which of the functions in Problems 1–16 are one-to-one?

1. $\{(1, 2), (2, 1), (3, 4), (4, 3)\}$
2. $\{(-1, 0), (0, 1), (1, -1), (2, 1)\}$
3. $\{(5, 4), (4, 3), (3, 3), (2, 4)\}$
4. $\{(5, 4), (4, 3), (3, 2), (2, 1)\}$

5.

Domain		Range
−2	→	−4
−1	→	−2
0	→	0
1	→	1
2	→	5

6.

Domain		Range
−2	→	−3
−1	→	−3
0	→	7
1	→	9
2	→	9

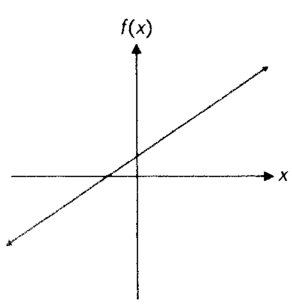
7.

Domain		Range
1	→	7
2	→	7
3	→	7
4	→	7
5	→	7

8.

Domain		Range
1	→	5
2	→	3
3	→	1
4	→	2
5	→	4

9.



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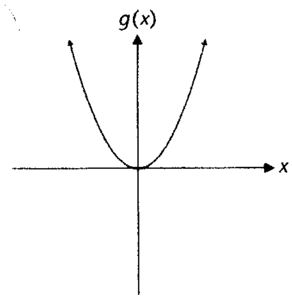

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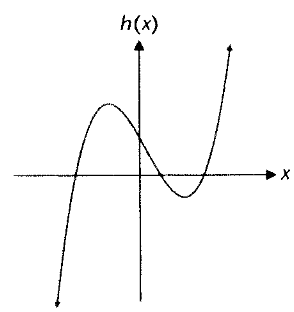

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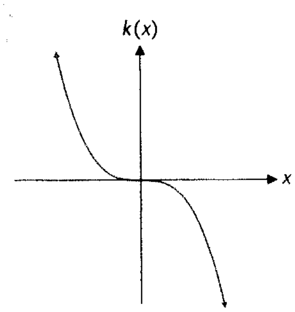

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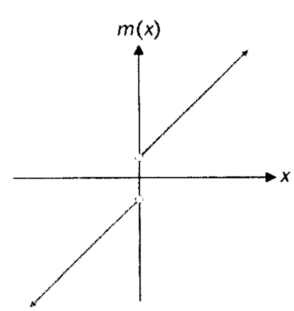

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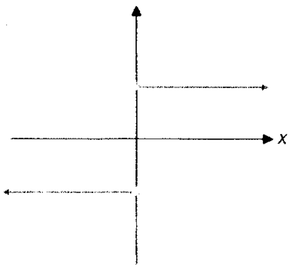

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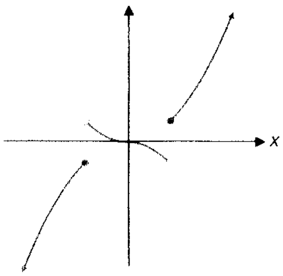

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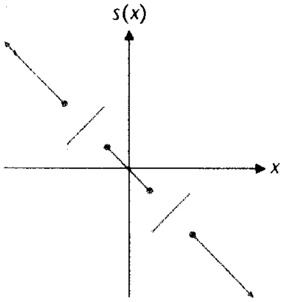


B

Which of the functions in Problems 17–22 are one-to-one?

17. $F(x) = \frac{1}{2}x + 2$ **18.** $G(x) = -\frac{1}{3}x + 1$

19. $H(x) = 4 - x^2$ **20.** $K(x) = \sqrt{4 - x}$

21. $M(x) = \sqrt{x + 1}$ **22.** $N(x) = x^2 - 1$

Problems 23–30 require the use of a graphing utility. Graph each function, and use the graph to determine if the function is one-to-one.

23. $f(x) = \dfrac{x^2 + |x|}{x}$ [redacted] $f(x) = \dfrac{x^2 - |x|}{x}$

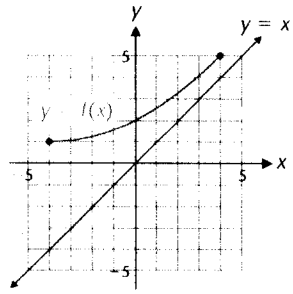
25. $f(x) = \dfrac{x^3 + |x|}{x}$ [redacted] $f(x) = \dfrac{|x|^3 + |x|}{x}$

27. $f(x) = \dfrac{x^2 - 4}{|x - 2|}$ [redacted] $f(x) = \dfrac{1 - x^2}{|x + 1|}$

29. $f(x) = \dfrac{x^3 - 9x}{|x^2 - 9|}$ **30.** $f(x) = \dfrac{4x - x^3}{|x^2 - 4|}$

In Problems 31–34, use the graph of the one-to-one function f to sketch the graph of f^{-1}. State the domain and range of f^{-1}.

31.



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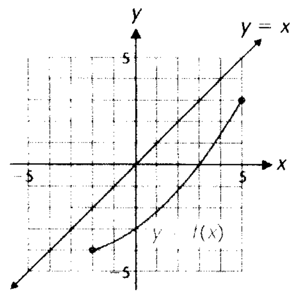

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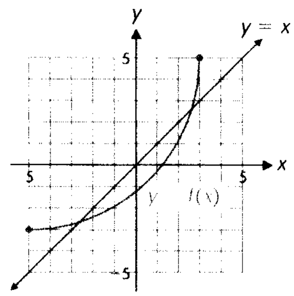

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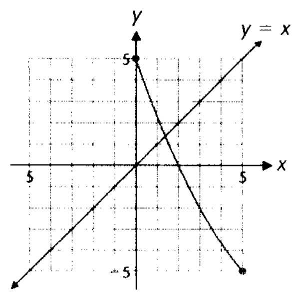


In Problems 35–40, verify that g is the inverse of the one-to-one function f by showing that $g[f(x)] = x$ and $f[g(x)] = x$. Sketch the graphs of f, g, and the line $y = x$ in the same coordinate system.

Check your graphs in Problems 35–40 by graphing f, g, and the line $y = x$ in a squared viewing window on a graphing utility.

35. $f(x) = 3x + 6$; $g(x) = \frac{1}{3}x - 2$

36. $f(x) = -\frac{1}{2}x + 2$; $g(x) = -2x + 4$

37. $f(x) = 4 + x^2, x \geq 0$; $g(x) = \sqrt{x - 4}$

38. $f(x) = \sqrt{x + 2}$; $g(x) = x^2 - 2, x \geq 0$

39. $f(x) = -\sqrt{x - 2}$; $g(x) = x^2 + 2, x \leq 0$

40. $f(x) = 6 - x^2, x \leq 0$; $g(x) = -\sqrt{6 - x}$

The functions in Problems 41–60 are one-to-one. Find f^{-1}.

41. $f(x) = \frac{1}{5}x$

42. $f(x) = 4x$

43. $f(x) = 2x + 7$

44. $f(x) = 0.25x + 2.25$

45. $f(x) = 0.2x + 0.4$

46. $f(x) = 7 - 8x$

47. $f(x) = 3 - \dfrac{2}{x}$

48. $f(x) = 5 + \dfrac{4}{x}$

49. $f(x) = \dfrac{2x}{x + 1}$

50. $f(x) = \dfrac{4x}{2 - x}$

51. $f(x) = \dfrac{0.2x - 0.4}{0.1x + 0.5}$

52. $f(x) = \dfrac{x - 0.2}{x + 0.5}$

53. $f(x) = 8x^3 - 5$

54. $f(x) = 2x^5 + 9$

55. $f(x) = 2 + \sqrt[5]{3x - 7}$

56. $f(x) = -1 + \sqrt[3]{4 - 5x}$

57. $f(x) = 2\sqrt{9 - x}$

58. $f(x) = 3\sqrt{x - 4}$

59. $f(x) = 2 + \sqrt{3 - x}$

60. $f(x) = 4 - \sqrt{x + 5}$

61. How are the x and y intercepts of a function and its inverse related?

62. Does a constant function have an inverse? Explain.

C

The functions in Problems 63–66 are one-to-one. Find f^{-1}.

63. $f(x) = (x - 1)^2 + 2, x \geq 1$

64. $f(x) = 3 - (x - 5)^2, x \leq 5$

65. $f(x) = x^2 + 2x - 2, x \leq -1$

66. $f(x) = x^2 + 8x + 7, x \geq -4$

The graph of each function in Problems 67–70 is one-quarter of the graph of the circle with radius 3 and center (0, 0). Find f^{-1}, find the domain and range of f^{-1}, and sketch the graphs of f and f^{-1} in the same coordinate system.

67. $f(x) = -\sqrt{9 - x^2}, 0 \leq x \leq 3$

68. $f(x) = \sqrt{9 - x^2}, 0 \leq x \leq 3$

69. $f(x) = \sqrt{9 - x^2}, -3 \leq x \leq 0$

70. $f(x) = -\sqrt{9 - x^2}, -3 \leq x \leq 0$

The graph of each function in Problems 71–74 is one-quarter of the graph of the circle with radius 1 and center (0, 1). Find f^{-1}, find the domain and range of f^{-1}, and sketch the graphs of f and f^{-1} in the same coordinate system.

71. $f(x) = 1 + \sqrt{1 - x^2}, 0 \leq x \leq 1$

72. $f(x) = 1 - \sqrt{1 - x^2}, 0 \leq x \leq 1$

73. $f(x) = 1 - \sqrt{1 - x^2}, -1 \leq x \leq 0$

74. $f(x) = 1 + \sqrt{1 - x^2}, -1 \leq x \leq 0$

75. Find $f^{-1}(x)$ for $f(x) = ax + b, a \neq 0$.

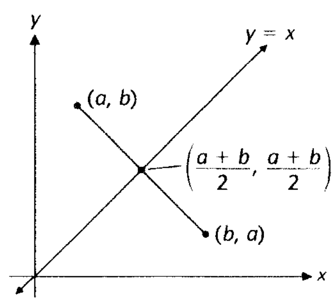
76. Find $f^{-1}(x)$ for $f(x) = \sqrt{a^2 - x^2}, a > 0, 0 \leq x \leq a$.

77. Refer to Problem 75. For which a and b is f its own inverse?

78. How could you recognize the graph of a function that is its own inverse?

79. Show that the line through the points (a, b) and (b, a), $a \neq b$, is perpendicular to the line $y = x$ (see the figure).

80. Show that the point $((a + b)/2, (a + b)/2)$ bisects the line segment from (a, b) to (b, a), $a \neq b$ (see the figure).



In Problems 81–84, the function f is not one-to-one. Find the inverses of the functions formed by restricting the domain of f as indicated.

Check Problems 81–84 by graphing f, g, and the line $y = x$ in a squared viewing window on a graphing utility. [Hint: To restrict the graph of $y = f(x)$ to an interval of the form $a \leq x \leq b$, enter $y = f(x)/((a \leq x)*(x \leq b))$.]

81. $f(x) = (2 - x)^2$:
(A) $x \leq 2$ (B) $x \geq 2$

82. $f(x) = (1 + x)^2$:
(A) $x \leq -1$ (B) $x \geq -1$

83. $f(x) = \sqrt{4x - x^2}$:
(A) $0 \leq x \leq 2$ (B) $2 \leq x \leq 4$

84. $f(x) = \sqrt{6x - x^2}$:
(A) $0 \leq x \leq 3$ (B) $3 \leq x \leq 6$