Example 2.11 For $X \sim \text{Gam}(\alpha, \lambda)$,

$$\mathbb{E}X = \frac{\alpha}{\lambda}, \ \text{Var}\, X = \frac{\alpha}{\lambda^2}. \tag{2.63}$$

In fact,

$$\begin{aligned}\mathbb{E}X &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x x^{\alpha-1} \mathrm{e}^{-\lambda x} \mathrm{d}x = \frac{1}{\lambda \Gamma(\alpha)} \int_0^\infty x^\alpha \mathrm{e}^{-x} \mathrm{d}x \\ &= \frac{\Gamma(\alpha+1)}{\lambda \Gamma(\alpha)} = \frac{\alpha}{\lambda}.\end{aligned}$$

Next,

$$\begin{aligned}\frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^2 x^{\alpha-1} \mathrm{e}^{-\lambda x} \mathrm{d}x &= \frac{1}{\lambda^2 \Gamma(\alpha)} \int_0^\infty x^{\alpha+1} \mathrm{e}^{-x} \mathrm{d}x \\ &= \frac{\Gamma(\alpha+2)}{\lambda^2 \Gamma(\alpha)} = \frac{(\alpha+1)\alpha}{\lambda^2}.\end{aligned}$$

This gives $\text{Var}\, X = (\alpha+1)\alpha/\lambda^2 - \alpha^2/\lambda^2 = \alpha/\lambda^2$. ■

For the above example, Find the optimal estimators of n/λ and n/λ^2

Ether UMVUE or MVB may be used to find the estimator.