6. Portfolio Theory and CAPM
(Capital Asset Pricing Model)

Key Assumptions of CAPM are

1. Household-investors have Mean-Variance preferences

2. All income in second period is derived from asset holdings; no other endowment is involved

3. Other assumptions to be discussed later.

6.1 Mean Variance Preferences

We have seen these preferences in previous lectures.

1 Mean variance preferences

- A quadratic von Neumann-Morgenstern utility depends only on the mean and the variance of the uncertain consumption. It takes the form

$$U(c_0, c_1, \ldots, c_S) = c_0 - \frac{1}{a} c_0^2 + \sum_{s=1}^{S} \pi_s (c_s - \frac{1}{a} c_s^2)$$

- Parameter a must be big enough so utility *decreases as the variance increases* and *increases as the mean increases*: the agent likes a bigger mean and dislikes a larger variance.

- Using the definition of mean and variance

$$U(c_0, c_1, \ldots, c_S) = (c_0 - \frac{1}{a} c_0^2) + E(c) - \frac{1}{a} E(c^2)$$

Equivalently

$$U(c_0, c_1, \ldots, c_S) = c_0 - \frac{1}{a} c_0^2 + E(c) - \frac{\text{var}(c) + E(c)^2}{a}$$

the household cares only about mean and variance of its consumption.

Mean variance preferences (cont)

- Recall the second assumption. Suppose there are n securities in the market with return

$$r_1, r_2, ,.., r_n$$

- The household forms a portfolio P of securities with proportions

$$w_1, w_2, ,.., w_n \quad , \quad \sum_{i=1}^{n} w_i = 1$$

Denote the return on this portfolio by

$$r_P = \sum_{i=1}^{n} w_i r_i .$$

- Remember, security payoffs vary over the states. Think of the r_i as varying over states $s = 1, 2, \ldots S$
- Denote date 0 invested assets by K (i.e. capital). Then, second period consumption is

$$\mathbf{c = K \sum_{i=1}^{n} w_i (1 + r_i) = K(1 + r_p).}$$

Hence

$$\mathbf{E(c) = K \sum_{i=1}^{n} w_i (1 + E(r_i)) \equiv K(1 + E(r_P))}$$

$$\mathbf{Var(c) = K^2 Var\left(\sum_{i=1}^{n} w_i (1 + r_i) \right) = K^2 var(r_P)}$$

Mean variance preferences (cont)

Now plug this into the utility function in (1) to derive

$$U(c_0, c_1, \ldots, c_S) = c_0 - \frac{1}{a}c_0^2 + K(1 + E(r_P) -$$

(2)

$$- \frac{K^2 var(r_P) + K^2(1 + E(r_P))^2}{a}$$

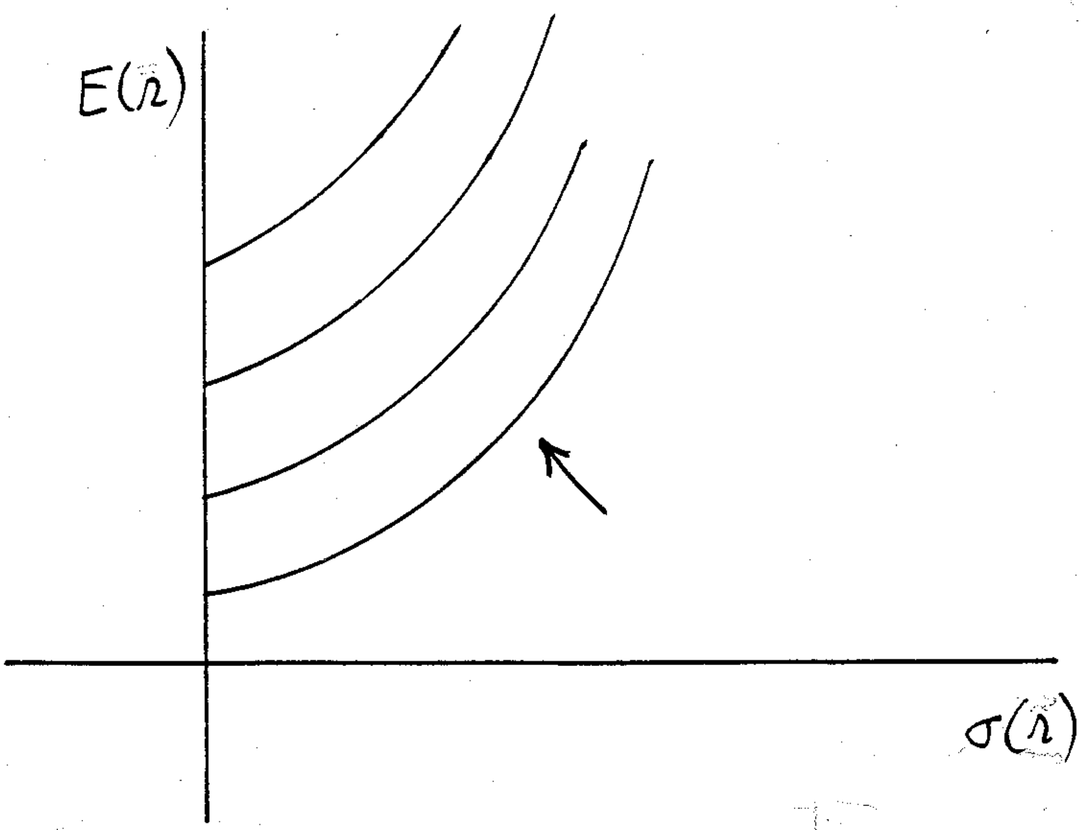
Given consumption c_0 and investment K, the utility function depends only upon Expected return and Variance of the portfolio.

- Very useful to use the standard deviation where

$$SD(x) = \sqrt{var(x)}$$

- CAPM theory ignores choice of c_0 and K
- We studied these already: choice of consumption and investment in the first period.
- We take c_0 and K as given and focus on the portfolio.
- Big advantage of Mean-Variance approach: does not depend upon number of states which is very large.
- In the next Figure we **draw the indifference curves of the utility function in (2)**

Mean-Standard Deviation Utility
Draw Indifference Curves



The graph justifies the procedure followed later, of selecting the efficient frontier of portfolios with **minimum standard deviation for each expected rate of return.**

6.2 Diversification

Recall Some Probability Concepts

- for two random variables x and y,

$$E(x + y) = E(x) + E(y).$$

- Expected payoff of a stock portfolio equals the sum of the expected payoff of the stocks.
- Variance of (x + y) satisfies an important relationship

$$\text{var}(x + y) = \text{var}(x) + \text{var}(y) + 2\text{cov}(x , y)$$

- Also $\text{var}(\frac{1}{2}x) = \frac{1}{4}\text{var}(x)$

Suppose two stocks have

$$q_1 = q_2 \ , \ E(r_1) = E(r_2), \ \text{Var}(r_1) = \text{Var}(r_2) \ , \text{Cov}(r_1 , r_2) = 0$$

- An investor would be indifferent between buying one unit of stock 1 and one unit of stock 2.

- But a portfolio $p = (\frac{1}{2},\frac{1}{2})$ has return $r_p = \frac{1}{2}r_1 + \frac{1}{2}r_2$ with

$$\text{Var}(r_p) = \text{Var}(\frac{1}{2}r_1 + \frac{1}{2}r_2) = \frac{1}{4}\text{Var}(r_1) + \frac{1}{4}\text{Var}(r_2) = \frac{1}{2}\text{Var}(r_i)$$

$$E(r_p) = \frac{1}{2}E(r_1) + \frac{1}{2}E(r_2) = E(r_i)$$

This is a basic argument for diversification

Diversification (contd.)

Consider a portfolio with weights w_i and where the rates of return on asset i is r_i. We denote moments of the rate of return r on the portfolio by:

$$r(s) = \sum_{i=1}^{n} w_i r_i(s)$$

$$E(r) = \sum_{i=1}^{n} w_i E(r_i)$$

$$var(r) = \sum_{s=1}^{S} \pi(s)[r(s) = E(r)]^2.$$

<u>When n = 2 we have</u>

$$Var(r) = \sum_{s=1}^{S} r(s)[w_1 r_1(s) + w_2 r_2(s) - w_1 E(r_1) - w_2 E(r_2)]^2$$

Diversification (contd.)

Hence,

$$\begin{aligned}\mathrm{Var}(r) &= \sum_{s=1}^{S} \pi(s)[w_1(r_1(s) - E(r_1)) + w_2(r_2(s) - E(r_2))]^2 \\ &= w_1^2 \sum_{s=1}^{S} \pi(s)[r_1(s) - E(r_1)]^2 + w_2^2 \sum_{s=1}^{S} \pi(s)[r_2(s) - E(r_2)]^2 \\ &+ 2w_1 w_2 \sum_{s=1}^{S} \pi(s)[r_1(s) - E(r_1)][r_2(s) - E(r_2)] \\ &= w_1^2 \mathrm{var}(r_1) + w_2^2 \mathrm{var}(r_2) + 2w_1 w_2 \mathrm{Cov}(r_1, r_2).\end{aligned}$$

Portfolio variance is the sum of matrix elements

$$\begin{bmatrix} w_1 w_1 \mathrm{Var}(r_1) & \Big| & w_1 w_2 \mathrm{Cov}(r_1, r_2) \\ ------ & | & ------- \\ w_1 w_2 \mathrm{Cov}(r_1, r_2) & \Big| & w_2 w_2 \mathrm{Var}(r_2) \end{bmatrix}$$

Diversification (contd.)

For a 4 asset portfolio the portfolio variance is the sum of the matrix elements

$$\begin{bmatrix} w_1 w_1 \mathrm{Var}(r_1) & w_1 w_2 \mathrm{Cov}(r_1, r_2) & w_1 w_3 \mathrm{Cov}(r_1, r_3) & w_1 w_4 \mathrm{Cov}(r_1, r_4) \\ w_2 w_1 \mathrm{Cov}(r_2, r_1) & w_2 w_2 \mathrm{Var}(r_2) & w_2 w_3 \mathrm{Cov}(r_2, r_3) & w_2 w_4 \mathrm{Cov}(r_2, r_4) \\ w_3 w_1 \mathrm{Cov}(r_3, r_1) & w_3 w_2 \mathrm{Cov}(r_3, r_2) & w_3 w_3 \mathrm{Var}(r_3) & w_3 w_4 \mathrm{Cov}(r_3, r_4) \\ w_4 w_1 \mathrm{Cov}(r_4, r_1) & w_4 w_2 \mathrm{Cov}(r_4, r_2) & w_4 w_3 \mathrm{Cov}(r_4, r_3) & w_4 w_4 \mathrm{Var}(r_4) \end{bmatrix}$$

We then define

$$\sigma_r = \sqrt{\mathrm{var}}(r)$$

Diversification with n = 2

Define the correlation coefficient by

$$\rho = \frac{\text{cov}(r_1, r_2)}{\sigma_{r_1} \sigma_{r_2}} \quad , \quad -1 \le \rho \le 1 .$$

<u>Case 1</u>: $\rho = -1$

$$\text{Cov}(r_1, r_2) = - \sigma_{r_1} \sigma_{r_2}$$

and portfolio variance is

$$\text{Var}(r) = w_1^2 \sigma_{r_1}^2 + w_2^2 \sigma_{r_2}^2 - 2 w_1 w_2 \sigma_{r_1} \sigma_{r_2}$$

$$= (w_1 \sigma_{r_1} - w_2 \sigma_{r_2})^2 .$$

Hence

$$\sigma_r = | w_1 \sigma_{r_1} - w_2 \sigma_{r_2} | .$$

Zero Variance Portfolio if ρ = - 1

We can reduce the portfolio variance to zero by selecting

$$\frac{w_1}{w_2} = \frac{\sigma_{r_2}}{\sigma_{r_1}}$$

$$w_1 + w_2 = 1$$

$$\Rightarrow \qquad \frac{\sigma_{r_2}}{\sigma_{r_1}} w_2 + w_2 = 1$$

$$w_2 = \frac{1}{1 + \frac{\sigma_{r_2}}{\sigma_{r_1}}}$$

Hence

$$w_1 = \frac{\sigma_{r_2}}{\sigma_{r_1} + \sigma_{r_2}} \qquad w_2 = \frac{\sigma_{r_1}}{\sigma_{r_1} + \sigma_{r_2}}$$

Zero Variance Portfolio if $\rho = -1$ (cont.)

The expected return of this portfolio

$$E(r) = w_1 E(r_1) + w_2 E(r_2)$$
$$= \frac{\sigma_{r_2}}{\sigma_{r_1} + \sigma_{r_2}} E(r_1) + \frac{\sigma_{r_1}}{\sigma_{r_1} + \sigma_{r_2}} E(r_2).$$

Example: $E(r_1) = 10\% \quad \sigma_{r_1} = 10\%$

$E(r_2) = 5\% \quad \sigma_{r_2} = 5\%$

For zero risk portfolio we then have

$$w_1 = \frac{5}{15} = \frac{1}{3} \qquad w_2 = \frac{10}{15} = \frac{2}{3}$$

$$E(r) = 1/3\ (10\%) + 2/3\ (5\%) = 6.667\%$$

$$\sigma_r = 0$$

Diagram: On the (E(r),σ) Space

But now if we vary w_1

$$E(r) = w_1 E(r_1) + (1 - w_1)\, E(r_2)$$

$$\sigma_r = w_1\sigma_{r_1} - (1 - w_1)\sigma_{r_2} = w_1(\sigma_{r_1} + \sigma_{r_2}) - \sigma_{r_2}$$

They are both linear in w_1. For $\sigma_r \geq 0$ we have

$$w_1 \geq \frac{\sigma_{r_2}}{\sigma_{r1} + \sigma_{r2}}.$$

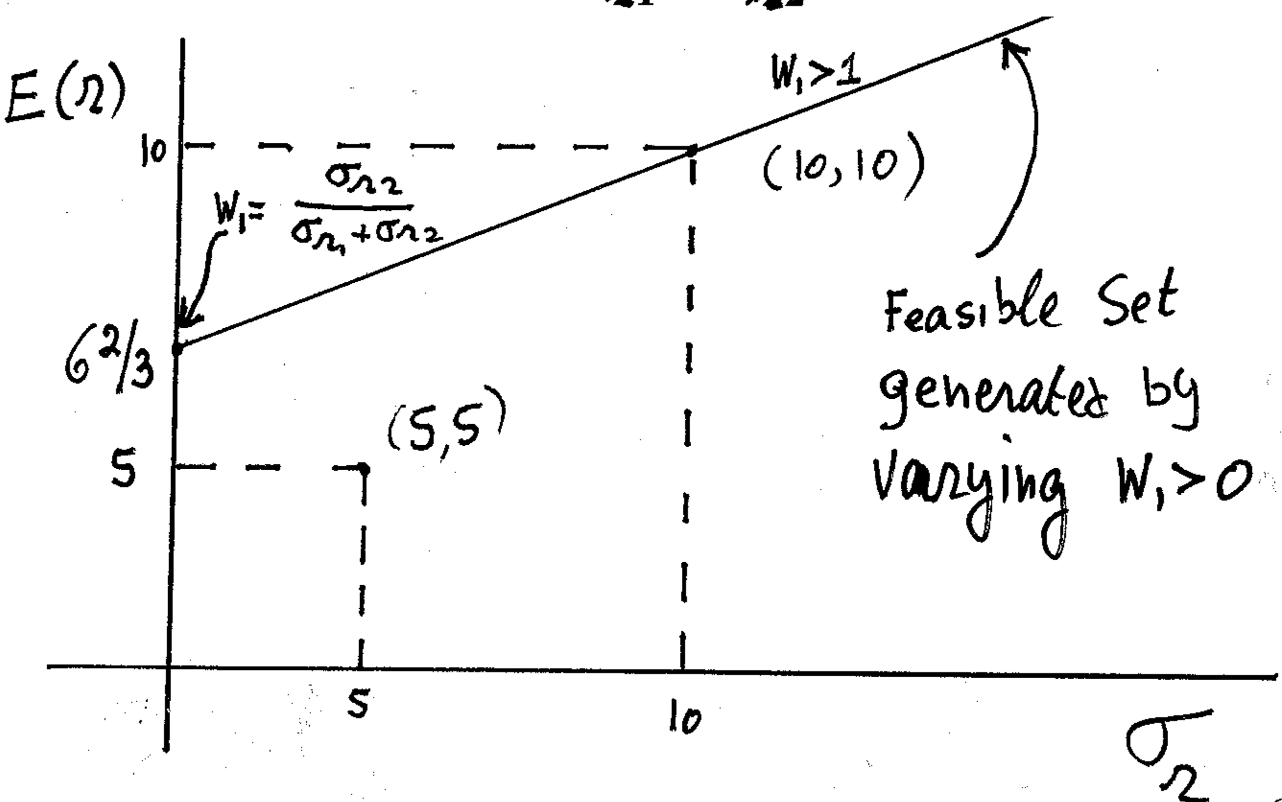
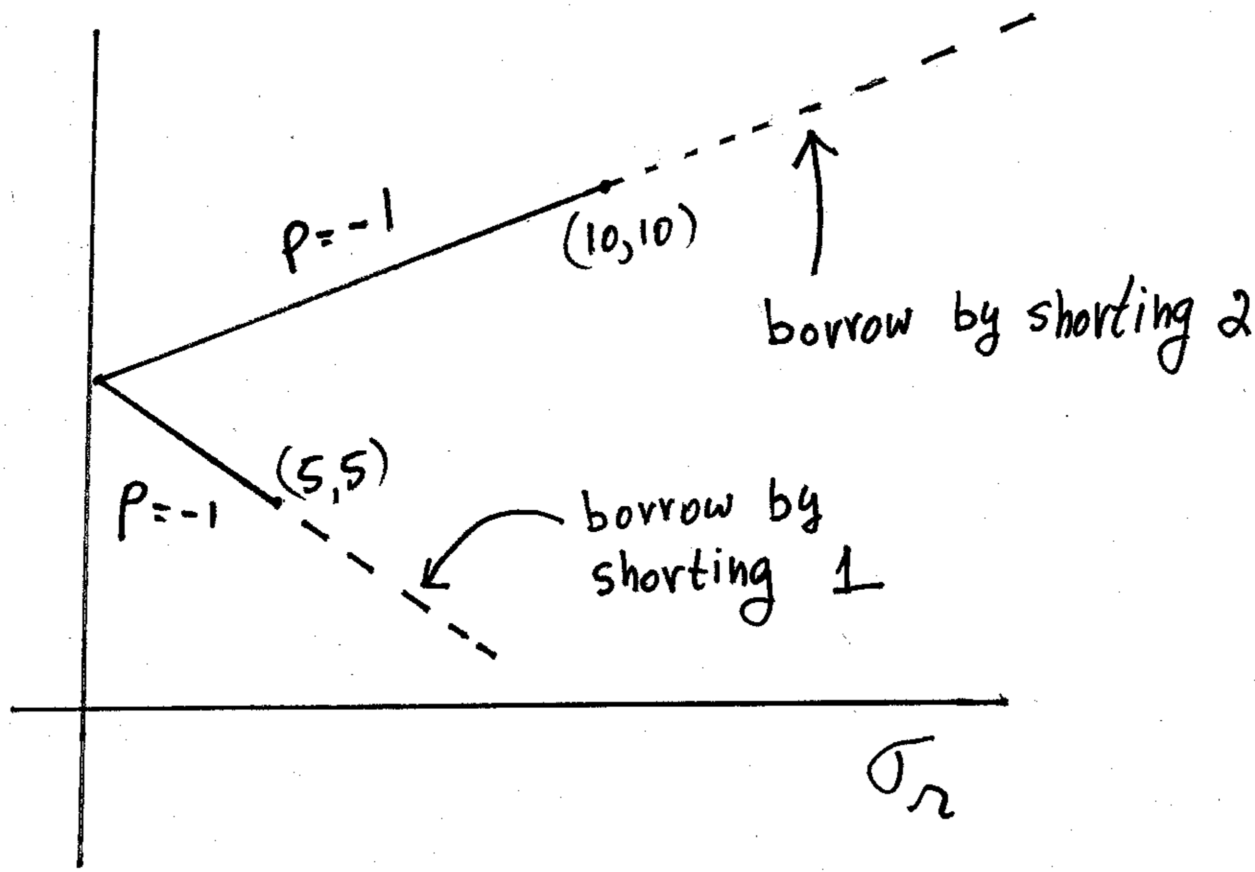


Diagram In (E(r),σ) Space (Cont.)

Suppose we also allow

$$w_1 < \frac{\sigma_{r_2}}{\sigma_{r1} + \sigma_{r2}}.$$

Then we includes the lower branch.
It is <u>inefficient</u>.



Case 2: $\rho = +1$

Then $Cov(r_1, r_2) = \sigma_{r_1}\sigma_{r_2}$ hence

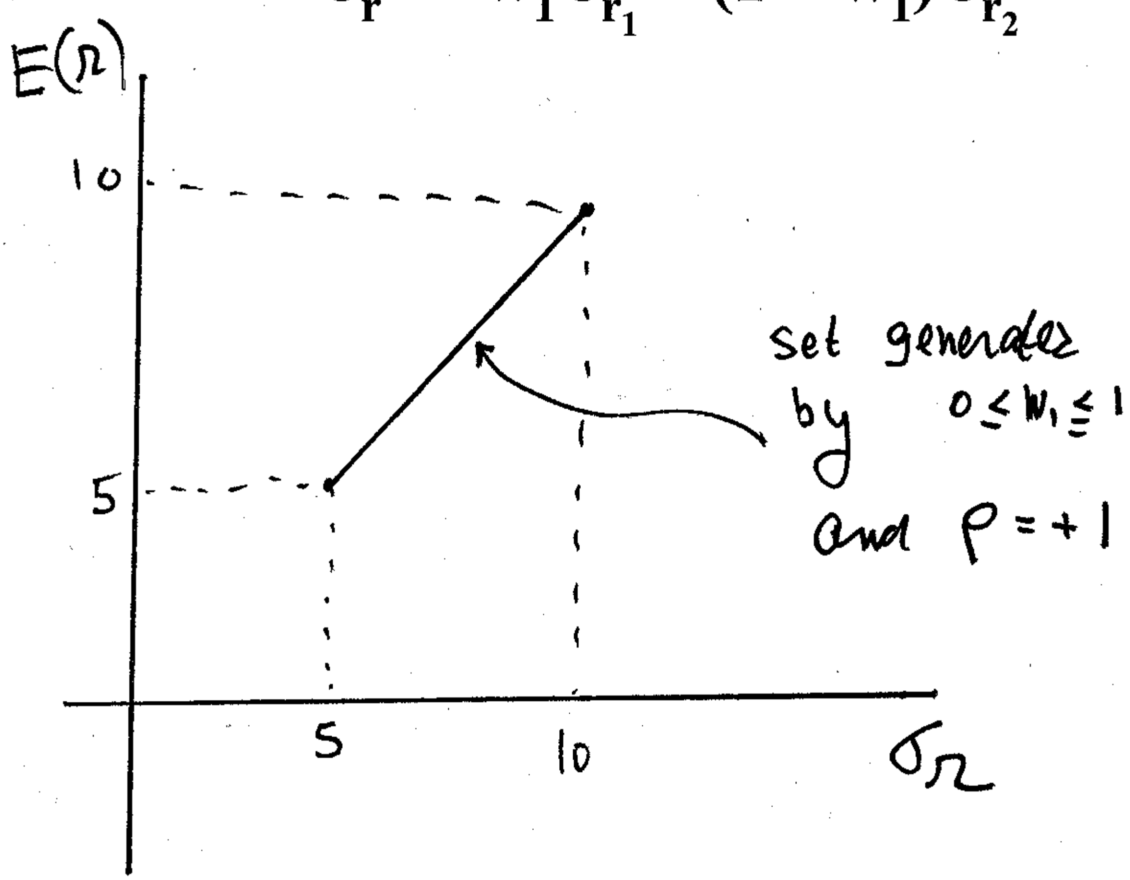
$$\sigma_r^2 = w_1^2\sigma_{r_1}^2 + w_2^2\sigma_{r_2}^2 + 2w_1w_2\sigma_{r_1}\sigma_{r_2}$$

$$= (w_1\sigma_{r_1} + w_2\sigma_{r_2})^2$$

Hence for $w_2 = 1 - w_1$

$$E(r) = w_1E(r_1) + (1 - w_1)E(r_2)$$

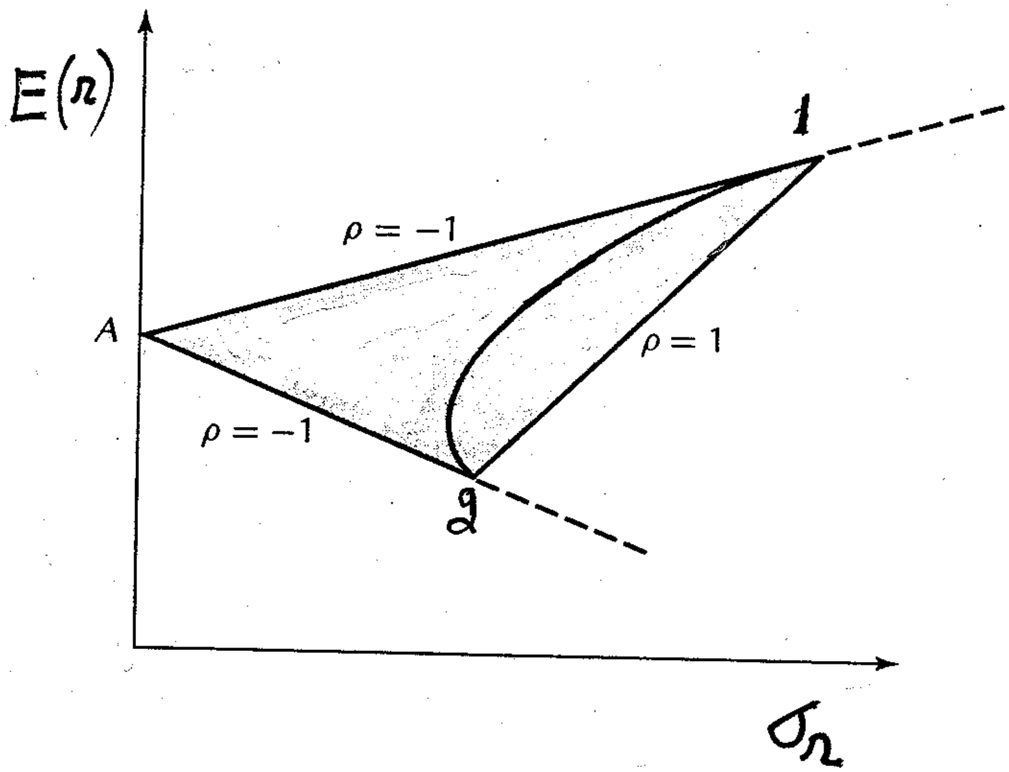
$$\sigma_r = w_1\sigma_{r_1} + (1 - w_1)\sigma_{r_2}$$



Diversification: The General n=2 Case

$$E(r) = w_1 E(r_1) + (1 - w_1) E(r_2)$$

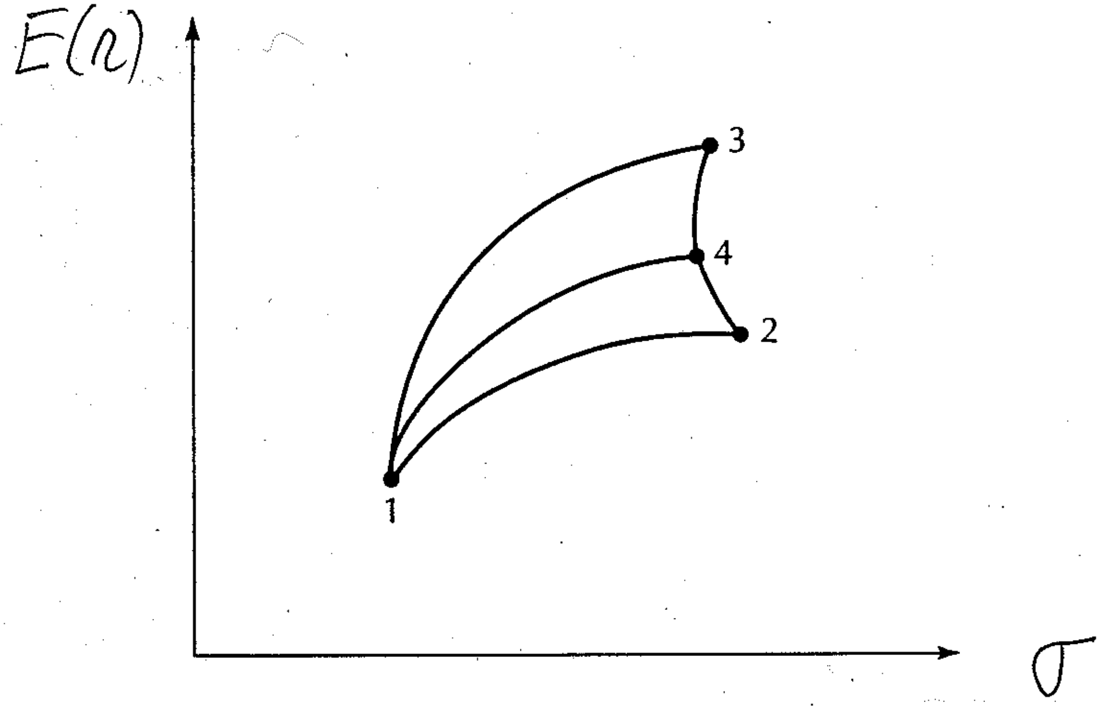
$$\sigma_r = \sqrt{w_1^2 Var(r_1) + w_2^2 Var(r_2) + 2 w_1 w_2 Cov(r_1, r_2)}$$



Solid curve = positive combinations
Dashed lines = shorting one asset

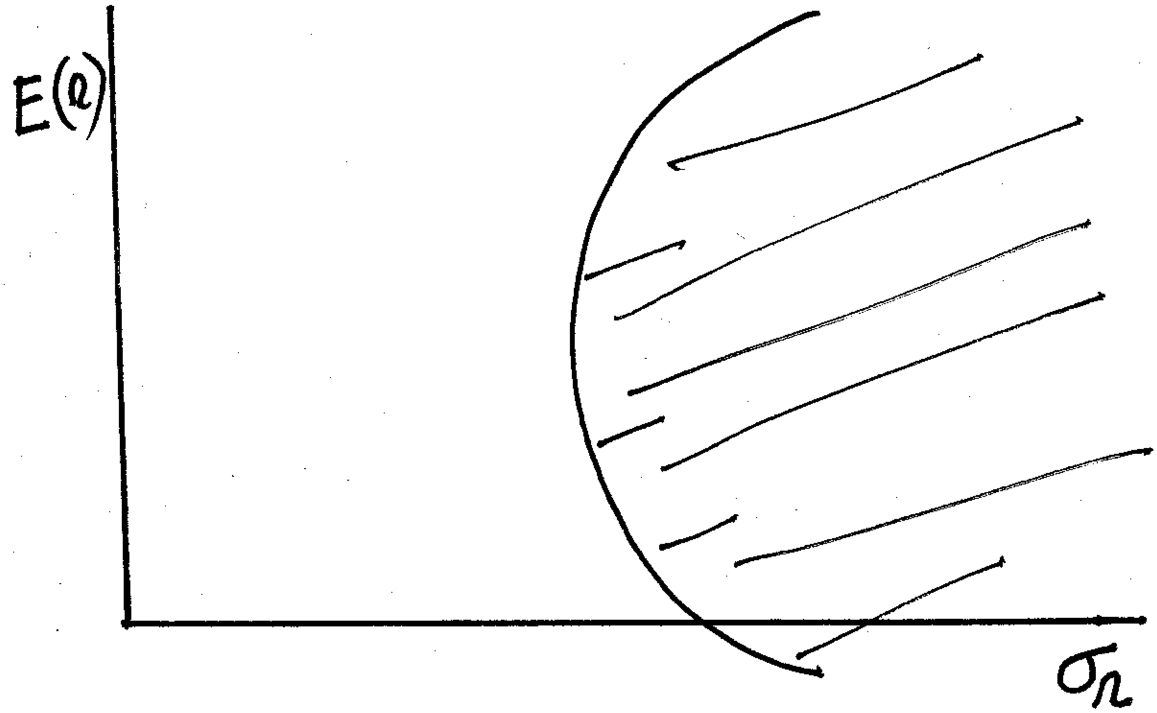
<u>Conclusion: Two Assets Trace A Feasible Curve</u>

Any Three Assets Trace A Region

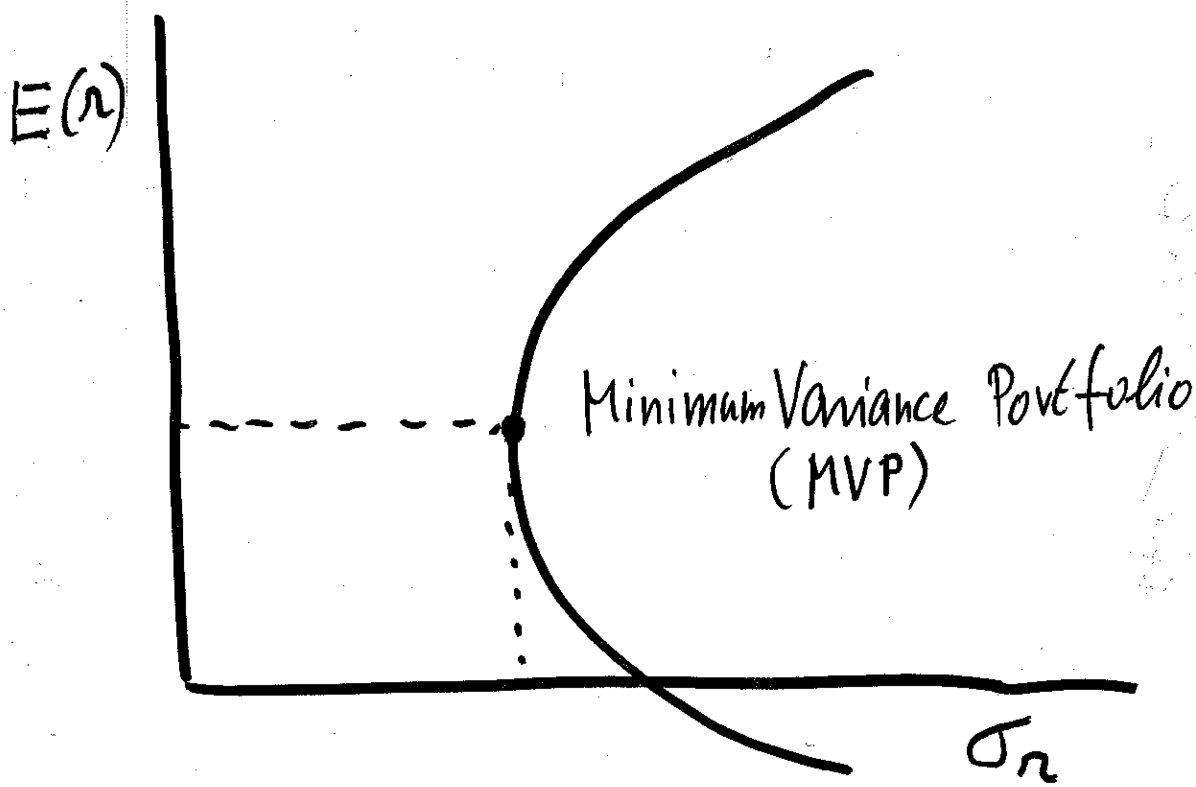


- Each pair (1-2), (1-3), (2-3) trace a curve
- Now Pick portfolio 4
- 4 is a portfolio generated by (2-3)
- (1 - 4) also trace a curve
- These portfolio curves then fill the region.

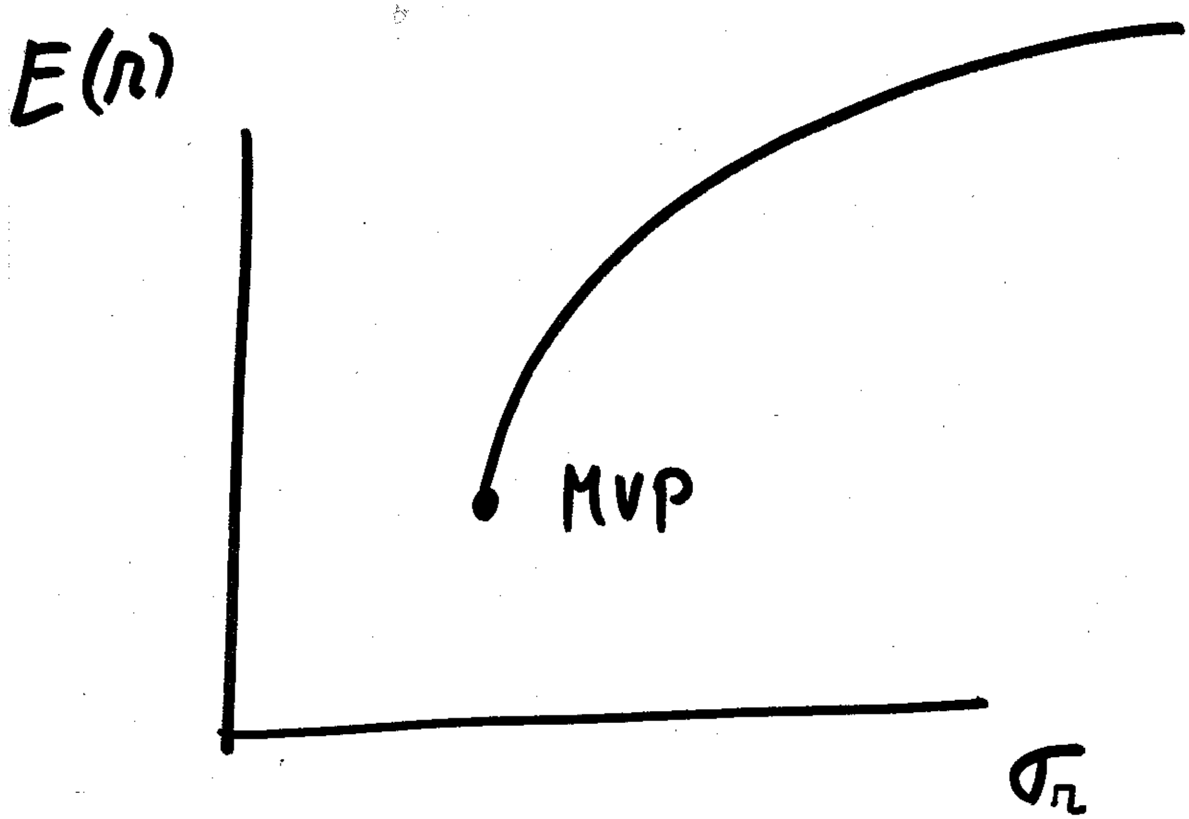
Many securities: A Convex Feasible Region



The Minimum Variance Curve



The Efficient Frontier of Risky Securities



For each value of E(r) The Frontier traces the portfolio with minimum variance.

<u>Note:</u> No riskless bond (yet)

Mathematical Definition of the Efficient Frontier

Given n rates of return $(r_1, r_2, r_3, \ldots r_n)$ of risky securities find a portfolio $(w_1, w_2, w_3, \ldots w_n)$ to minimize the variance of the portfolio subject to a given expected return:

$$\min_{(w_1, w_2, \ldots, w_n)} \quad Var(r) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

Subject to

$$\sum_{i=1}^{n} w_i E(r_i) = \text{specified } \bar{r}$$

$$\sum_{i=1}^{n} w_i = 1.$$

This is a standard linear programming problem. See Text BKM pp. 223-224.

$$r = \sum_{i=1}^{n} w_i r_i$$

Solution of the problem

Write down the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} + \lambda[\bar{r} - \sum_{i=1}^{n} w_i E(r_i)] + \mu[1 - \sum_{i=1}^{n} w_i]$$

with the first Order Conditions

$$\frac{\partial \mathcal{L}}{\partial w_i} = \sum_{j=1}^{n} w_j \sigma_{ij} - \lambda E(r_i) - \mu = 0 \quad i = 1, 2, 3, \ldots, n.$$

With the constraints they imply $n + 2$ equations

$$\sum_{j=1}^{n} w_j \sigma_{ij} - \lambda E(r_i) - \mu = 0 \quad i = 1, 2, 3, \ldots, n.$$

$$\sum_{i=1}^{n} w_i E(r_i) = \bar{r}$$

$$\sum_{i=1}^{n} w_i = 1.$$

in the $n+2$ unknowns $(w_1, w_2, w_3, \ldots w_n, \lambda, \mu)$.

Now vary $\bar{r}$. The family of solutions span the Efficient Frontier.

Solution

- **For each $\bar{r}$ you get a solution of one point.**

- **The frontier Traces all solutions for different $\bar{r}$.**

- **You get the efficient frontier by ignoring the lower portion of the curve.**

- **Remember, we have not introduced the bond.**

A Two Fund Theorem

Let $(w_1^1, w_2^1, w_3^1, \ldots w_n^1, \lambda^1, \mu^1)$ be a solution for r^1 and $(w_1^2, w_2^2, w_3^2, \ldots w_n^2, \lambda^2, \mu^2)$ a solution for r^2. The for any real number α the linear combination

$$\alpha \begin{pmatrix} w_1^1 \\ w_2^1 \\ \bullet \\ \bullet \\ \lambda^1 \\ \mu^1 \end{pmatrix} + (1 - \alpha) \begin{pmatrix} w_1^2 \\ w_2^2 \\ \bullet \\ \bullet \\ \lambda^2 \\ \mu^2 \end{pmatrix}$$

is a solution for $r = \alpha r^1 + (1 - \alpha) r^2$.

Meaning

- Any two members of the frontier could reproduce any other member of the frontier
- There are two funds such that any optimal portfolio is a linear combination of these funds.

6.3 Leverage

- It is not difficult to increase expected return: use the riskless bond
- You want to invest \$1
- Consider 'General Motors' (GM) stock. The riskless bond pays interest rate r_f (it costs q_B, $(1 + r_f) = 1/q_B$
- It is realistic to assume that $E(r_{GM}) > r_f$.

Leverage Strategy. **<u>At cost of \$1 create a new portfolio</u>**:

- Out of your \$1 put t dollars into GM and $(1 - t)$ dollars into the bond where t *may be negative* !
- Payoff is $R_{GM} \in \mathbb{R}^S$ for GM. The price of GM = q_{GM}; and the bond price is q_B.
- For t dollars, you buy $\mathbf{t/q_{GM}}$ shares of General Motors
- This yields a random profit of $\mathbf{tR_{GM}(s)/q_{GM}}$ in state s.
- The return on a dollar invested in General Motors in state s is the ratio $\mathbf{r_{GM}(s) = R_{GM}(s)/q_{GM}}$.

Leverage (Cont.)

Hence, your expected return from this portfolio is

$$E\left((1-t)r_f + t r_{GM}\right) = (1-t)r_f + tE(r_{GM})$$

$$= r_f + t[E(r_{GM}) - r_f].$$

Since $E(r_{GM}) > r_f$ we have

Expected Return of this investment strategy

- If you select large t so that $(1-t) < 0$ (i.e. you borrow) you sell the bond and buy GM (you are "short" the bond and "long" the stock) the following expression grows without bound as t becomes large

$$E\left((1-t)r_f + t r_{GM}\right) = = r_f + t[E(r_{GM}) - r_f]$$

Your expected return can be as large as you wish!!

But note that although the expected return of the new portfolio is

(1) $$E\left((1-t)r_f + t r_{GM}\right) = (1-t)r_f + tE(r_{GM})$$

what about the risk?

The risk of this investment strategy

- The bond has a return with no variance and zero covariance with any other asset.
- This introduces the possibilities of risk spreading.
- The variance of this new portfolio is

$$var\left((1 - t)r_f + tr_{GM}\right) = t^2 var(r_{GM}).$$

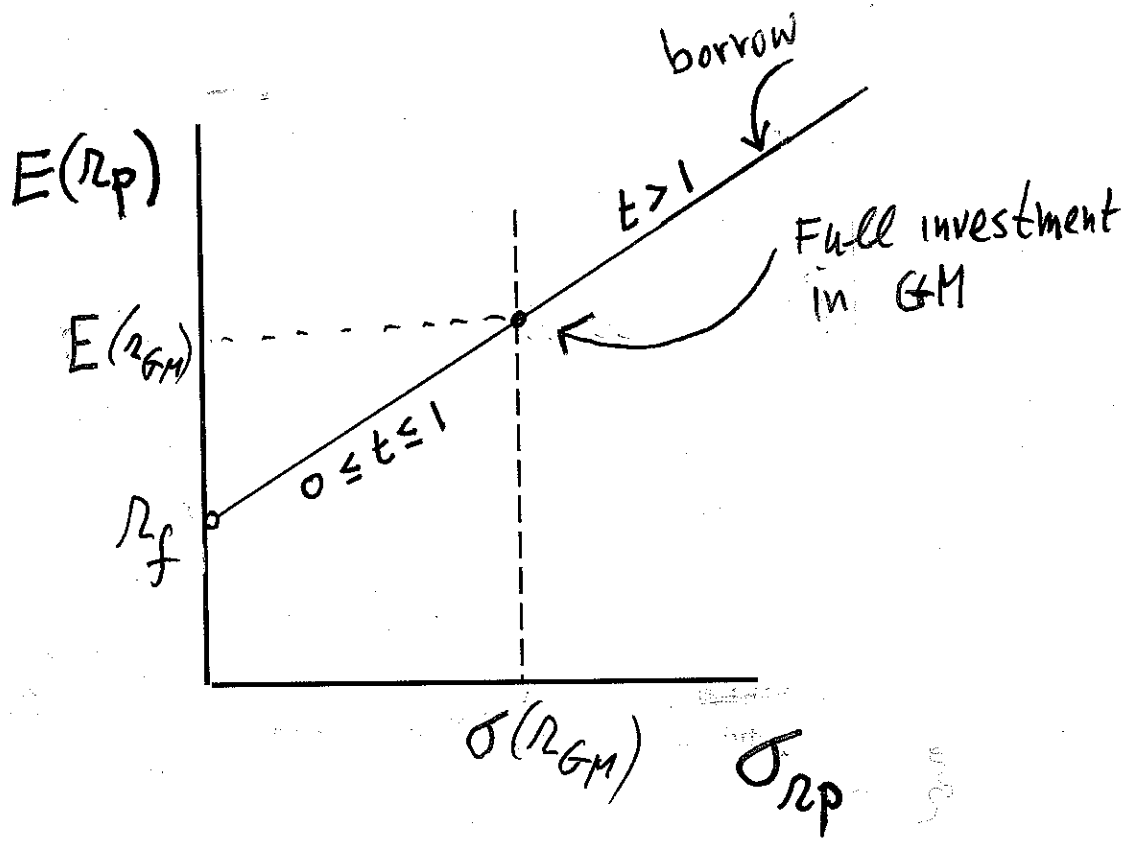
and

$$\text{(2)} \qquad SD\left((1 - t)r_f + tr_{GM}\right) = tSD(r_{GM}).$$

The expected return and standard deviation of the portfolio are Increasing and Linear in t!

- **We Trace out in the next figure the straight line of (expected return, standard deviation) described in equations (1)-(2).**
- Using the riskless bond an investor can attain any combination of expected return and standard deviation on the straight line connecting the returns of the stock GM to the bond.

The Effect of Leverage



The diagram traces the combinations of risk and return when a riskless bond is in the market to provide a riskless rate of r_f

Introducing a Risk Free Bond

- We generalize and introduce risk free bond.
- Take a portfolio of risky assets with expected return r.
- You invest $K in a new portfolio with return r_K composed as follows:

  proportion α invested in the risk free bond
  proportion $1 - \alpha$ in the given portfolio

The return on your portfolio must be

$$r_K = \alpha r_f + (1-\alpha) r$$

Where is your portfolio in the space of $(E(r), \sigma)$?

A direct calculation shows

$$E\left(\alpha r_f + (1-\alpha) r\right) = \alpha r_f + (1-\alpha) E(r)$$

$$SD\left(\alpha r_f + (1-\alpha) r\right) = (1-\alpha) SD(r).$$

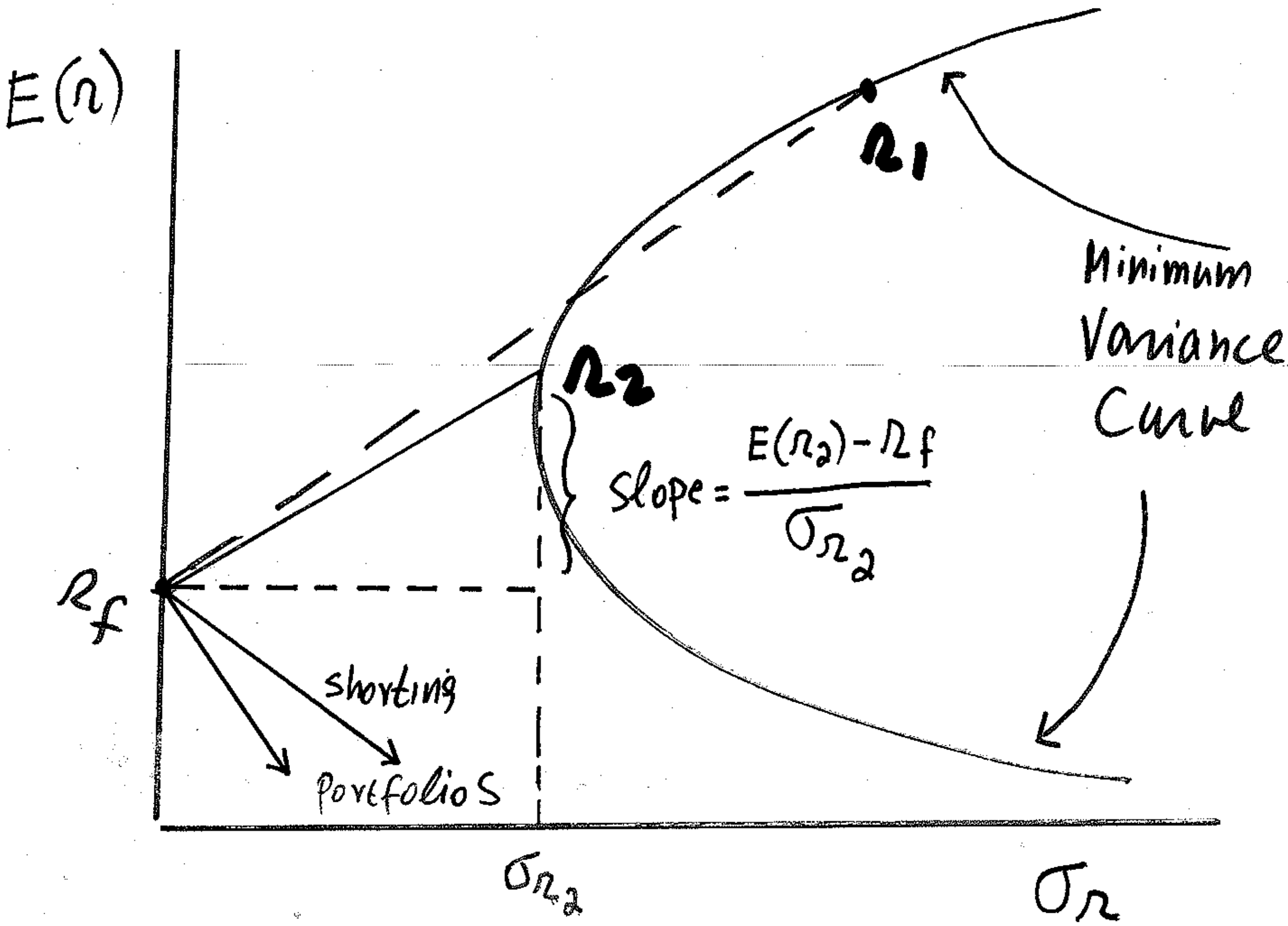
Introducing a Risk Free Bond (cont.)

Now vary the parameter α:

- **These two equations trace a line called the Capital Allocation Line (CAL) .**
- **If you select a different portfolio with a different expected return, you trace a new CAL.**
- **In the following Figure we trace the curve of minimum variance portfolios and several Capital Allocation lines which start from the risk free return r_f**

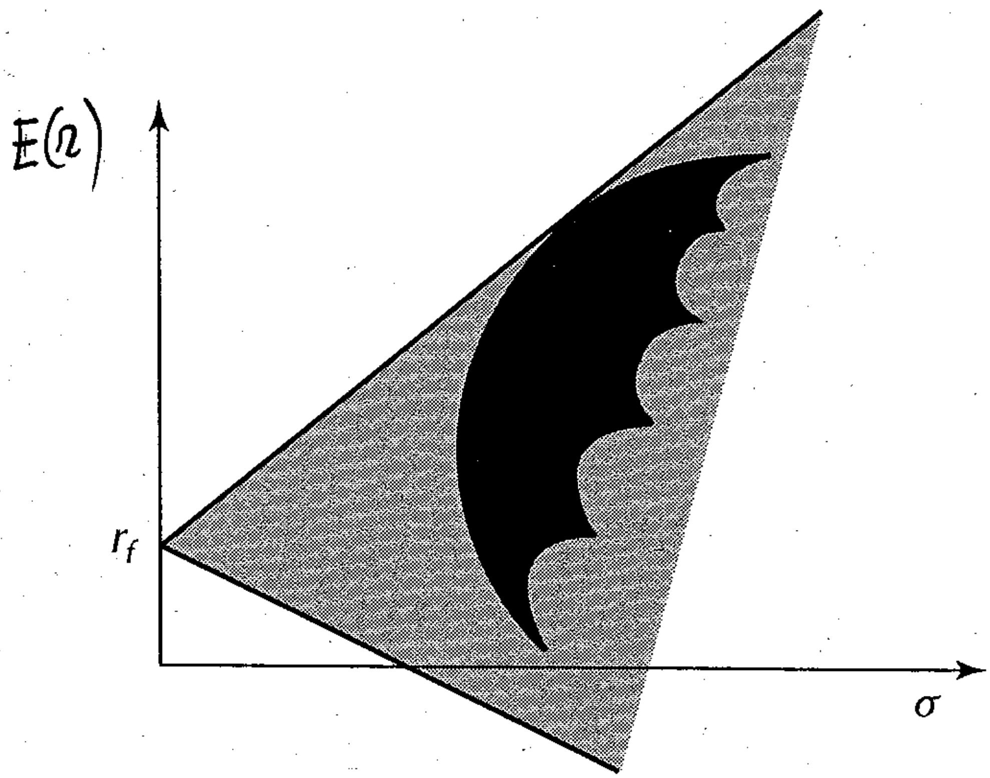
WITH RISK FREE BOND EVERY FRONTIER PORTFOLIO MUST BE ON SOME CAPITAL ALLOCATION LINE.

Risk Free Bond (contd.)



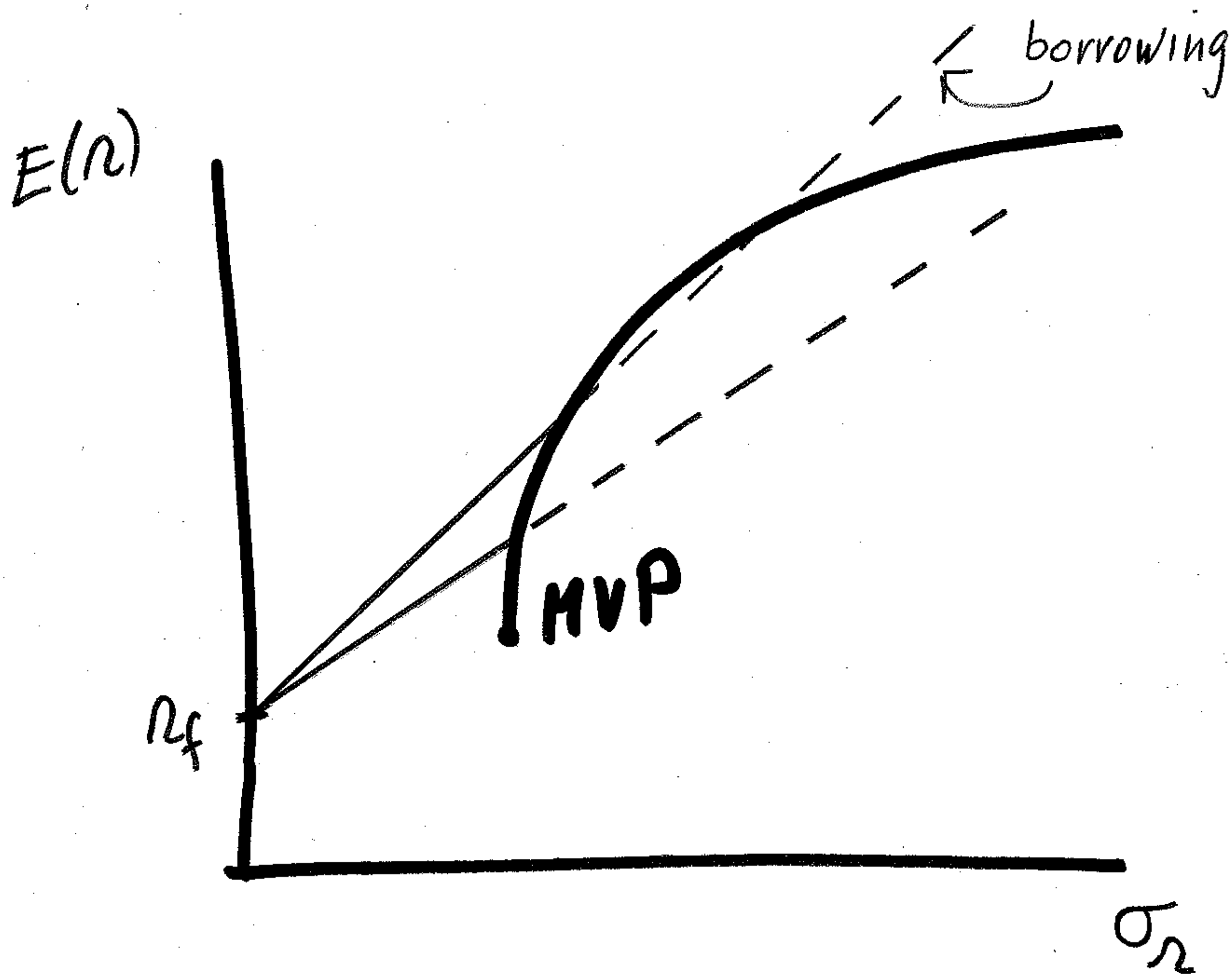
We draw two CAL from r_f to r_1 and r_2. These are all the portfolio payoff combinations that can be attained with leverage.

Effect of Risk Free Asset



Enlarging the feasible set by linear extension of every portfolio payoff via leverage of that portfolio!

Efficient Frontier and Risk Free Bond

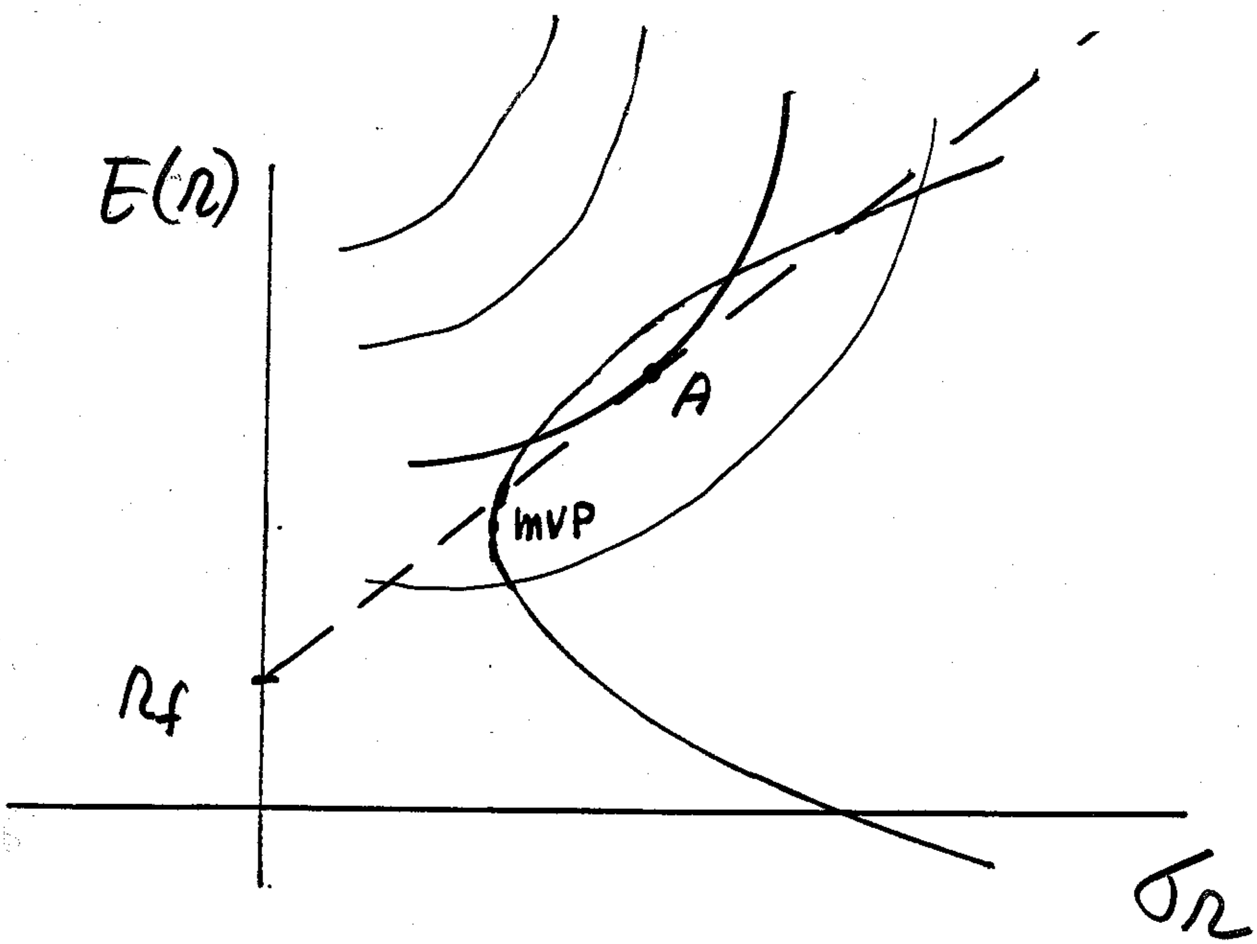


Using the Risk Free bond and the existing securities (r_1, r_2,..., r_n) each investor can select his own Capital Allocation Line.

<u>WHAT IS BEST?</u>

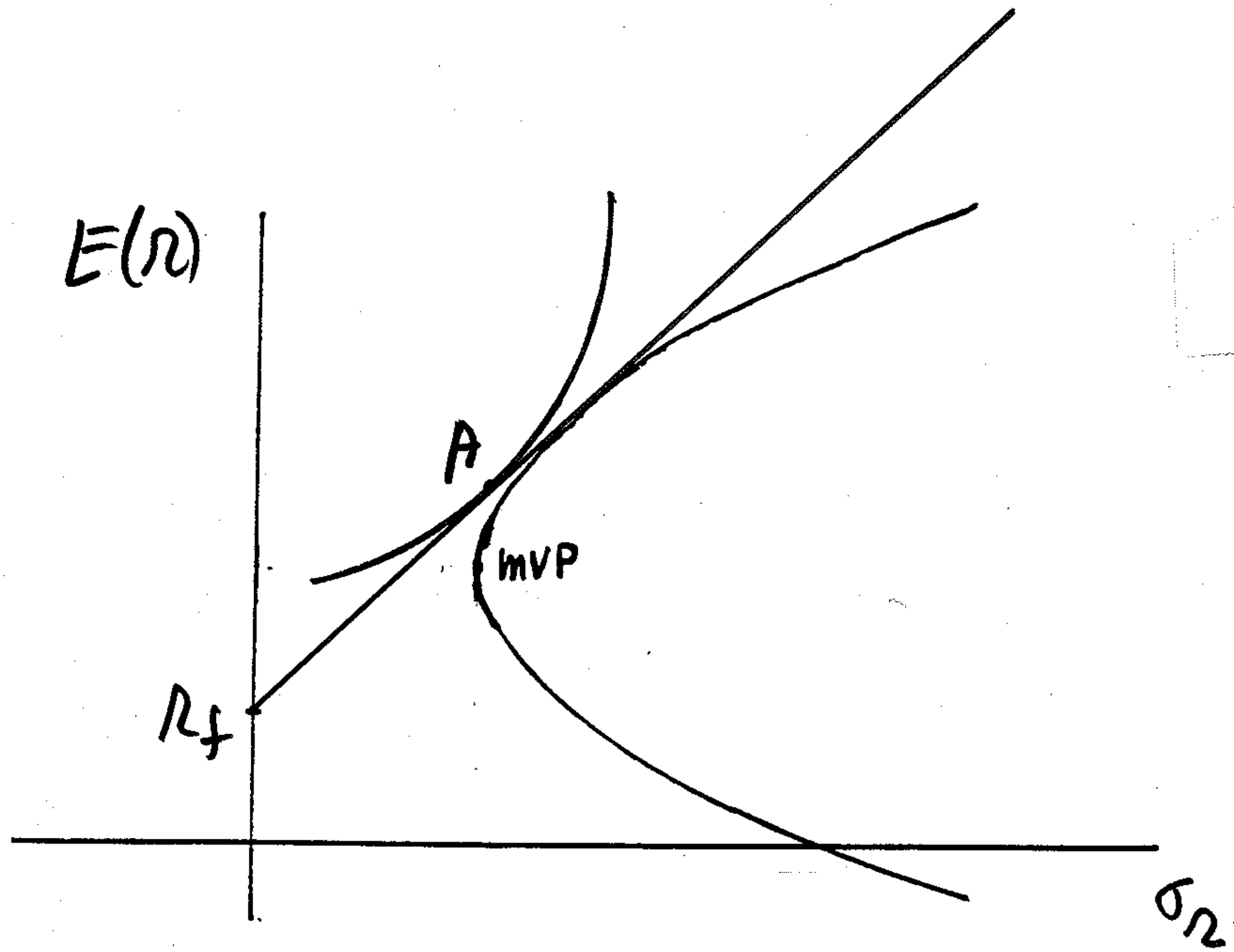
6.4 Optimal Investors Choice and Equilibrium

- **Given a market with n securities ($r_1, r_2,\ldots,r_n$) and a riskless bond paying r_f each investor selects his CAL to optimize**



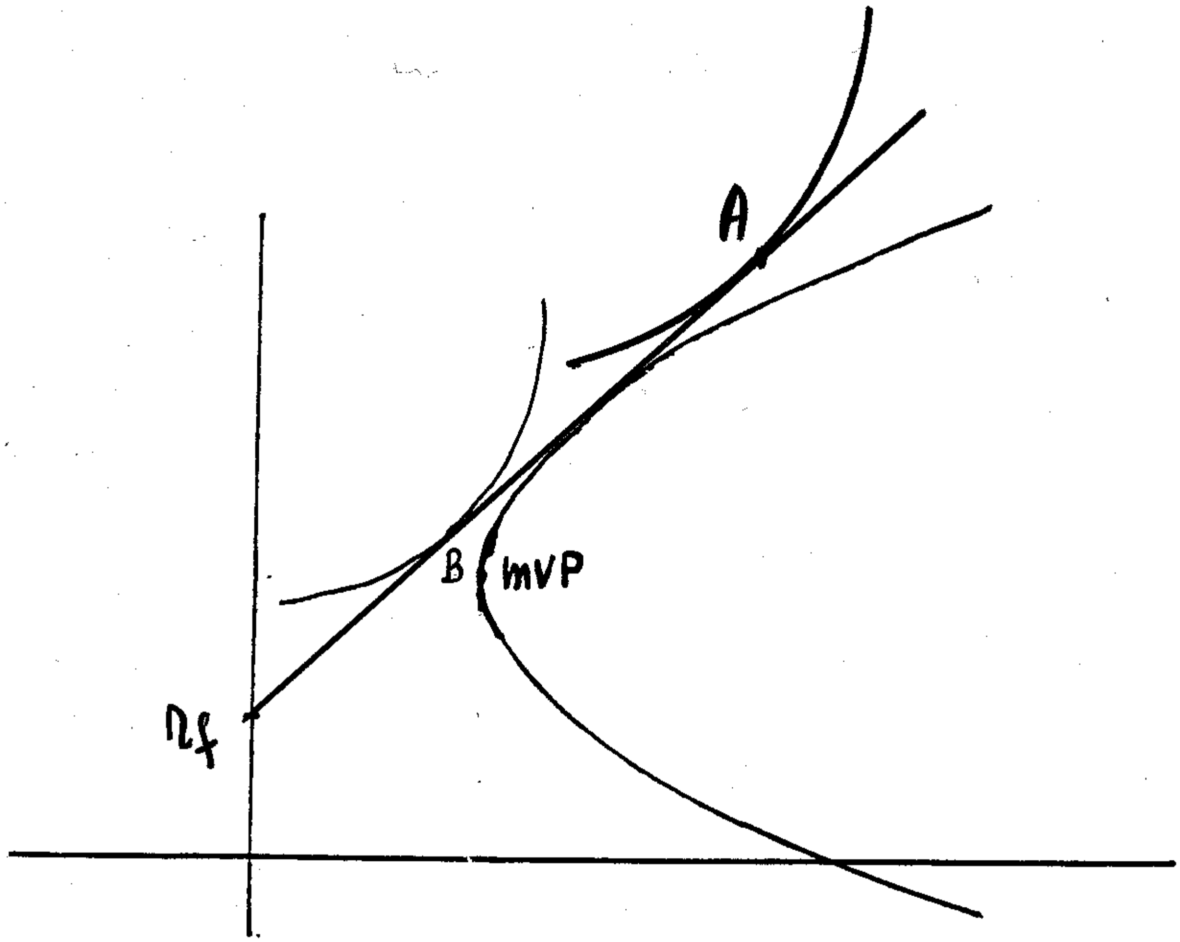
<u>Portfolio A cannot be an optimum!!</u>

Optimal Choice (cont.)



A is an optimum without borrowing.

Optimum Choice (cont.)



A is an optimal portfolio with leverage.

<u>Theorem</u>: Given a market, all investors will select optimal portfolios along the same line.

Sharpe Ratio

- **The problem: all diverse optimal portfolios are along the same line but in equilibrium the total supply of every security must be held by somebody**

- **Some reflection shows that the "Market Portfolio," consisting of the portfolio of all available risky securities, must also be located on that same line.**

- **This can be formulated in a simple way. Following William Sharpe define the Sharpe ratio of a portfolio θ with returns $r_\theta = \sum_{j=1}^{n} r_j \theta_j$ as**

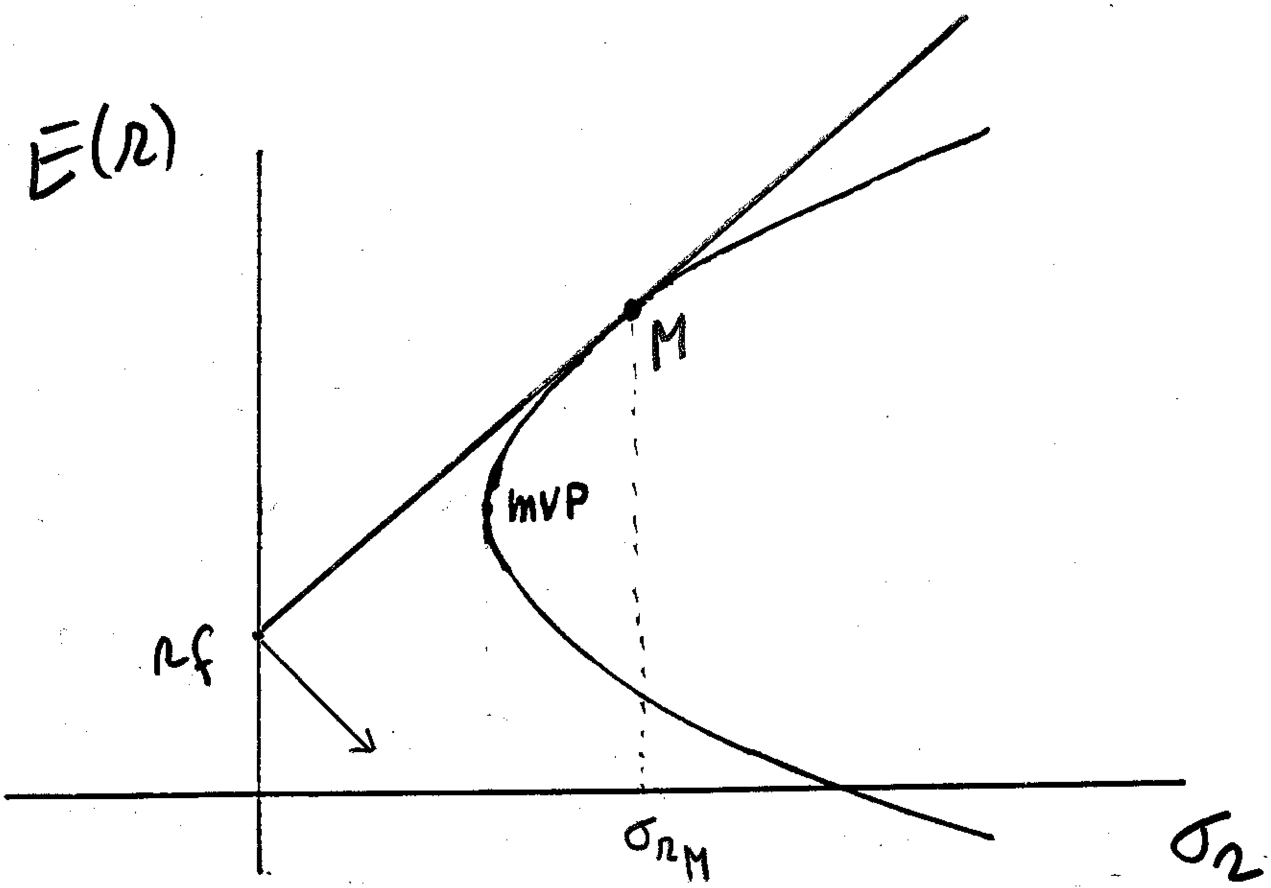
$$SR = \frac{E(r_\theta) - r_f}{SD(r_\theta)}.$$

The Sharp Ratio is the slope of a CAL connecting the portfolio θ to the rate r_f in the (expected return, standard deviation)-space.

Basic Observation: The market portfolio has the highest Sharp ratio and hence the market portfolio of risky assets is on the efficient frontier of risky securities.

Equilibrium: The Market Portfolio

(Market Price of Risk)



Equilibrium Capital Market Line:

- **The capital market line is defined by**

$$E(r) - r_f = \frac{E(r_m) - r_f}{\sigma_M}\sigma$$

- **All efficient portfolios including the bond are on the capital market line.**

Proof of Slope

Consider investing an additional proportion δ of the market portfolio r_M while borrowing at r_f. The resulting return would then be

$$r = (1 + \delta)r_M - \delta r_f$$

$$E(r) = E(r_M) + \delta[E(r_M) - r_f]$$

Hence $$\Delta E(r) = \delta[E(r_M) - r_f]$$

$$\sigma^2 = (1 + \delta)^2 \sigma_M^2$$

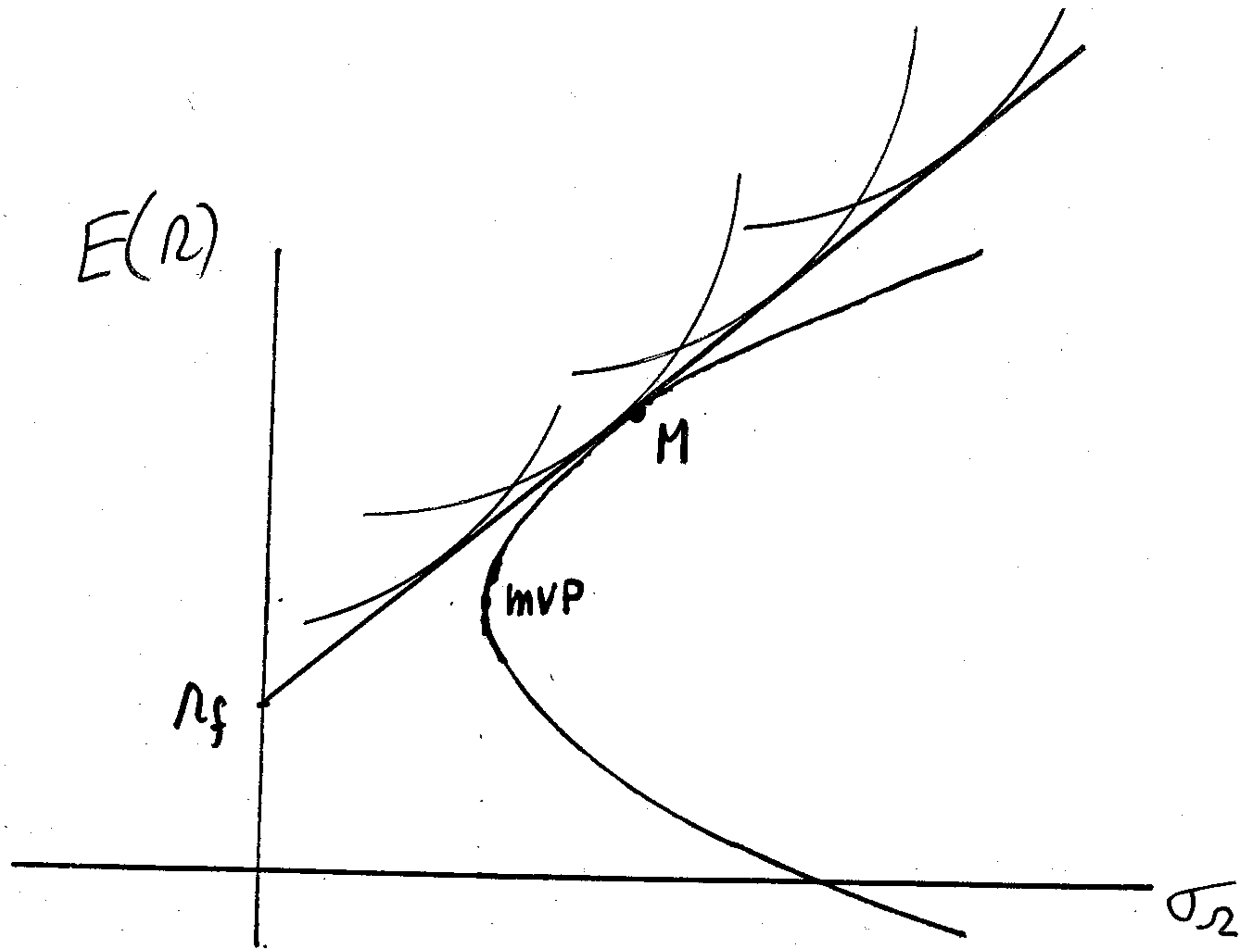
$$\sigma = (1 + \delta)\sigma_M$$

$$\Delta\sigma = \delta\sigma_M$$

Hence we prove that the slope must be

$$\frac{\Delta E(r)}{\Delta\sigma} = \frac{E(r_M) - r_f}{\sigma_M}.$$

Equilibrium Distribution of Portfolios



Two Fund Theorem: **In equilibrium all optimal portfolios are distributed along the lines consisting of linear combinations of the Market Portfolio and the risk free asset.**

CAPM - The Pricing Theory

- CAPM theory has deep implications to what constitutes risk and what is the return the market pays for taking risk.

- Risk as Covariance with the Market

Decomposition of a contribution of security

- Contribution to return is simply the expected return
-
- Contribution to total market risk (marginal riskiness) is more subtle.

- The marginal risk of a little additional security j can then be calculated as follows:

$$\frac{d}{dt}\text{var}(r_M + t\,r_j) = \frac{d}{dt}\left(\text{var}(r_M) + 2t\,\text{cov}(r_j, r_M) + t^2\,\text{var}(r_j)\right)$$

The Pricing Theory(Cont.)

When evaluated at t = 0 the result is

$$2\operatorname{cov}(r_j, r_M).$$

- **Hence, the marginal riskiness of a security is not measured by its own variance, but rather by its covariance with aggregate endowments.**

Theorem: **The expected return $E(r_i)$ of any security satisfies**

$$E(r_i) - r_f = \beta_i \, [E(r_m) - r_f]$$

where

$$\beta_i = \frac{\operatorname{Cov}(r_i, r_M)}{\sigma_M^2}$$

<u>Informal Proof</u>: using an arbitrage argument.

<u>Initial position: 100% in M</u>.

- **Now borrow a little δ and invest in M**

$$r = r_M + \delta[r_M - r_f] = (1 + \delta)r_M - \delta r_f$$

$$E(r) = E(r_M) + \delta[E(r_M) - r_f]$$

$$\Delta E(r) = \delta[E(r_M) - r_f]$$

Now compute the added risk to find that

$$\sigma_r^2 = (1 + \delta)^2 \sigma_M^2$$

$$= (1 + 2\delta + \delta^2)\sigma_M^2 .$$

since δ is small, ignore δ^2

$$\sigma_r^2 \approx (1 + 2\delta)\sigma_M^2$$

$$\Delta \sigma^2 = 2\delta \sigma_M^2$$

Hence

(*)

$$\frac{\Delta E(r)}{\Delta \sigma^2} = \frac{E(r_M) - r_f}{2\sigma_M^2}$$

- **Now borrow a little δ and invest in risky asset i.**

$$r = r_M + \delta[r_i - r_f]$$

$$\Delta E(r) = \delta[E(r_i) - r_f]$$

But the assessment of risk is

$$\sigma_r^2 = \sigma_M^2 + \delta^2\sigma_i^2 + 2\delta \mathrm{Cov}(r_i, r_M)$$

Again, ignoring δ^2 term to find that

$$\Delta\delta = \mathrm{Cov}(r_i, r_M)$$

Hence we find the return risk ratio to be

(**) $$\frac{\Delta E(r)}{\Delta\sigma^2} \approx \frac{E(r_i) - r_f}{2\mathrm{Cov}(r_i, r_M)}$$

Now compare (*) and (**).

In equilibrium arbitrage will ensure that the extra expected return ($\Delta E(r)$) per unit of risk added ($\Delta\sigma^2$), must be the same for all assets.

Hence

$$\frac{E(r_M - r_f)}{2\sigma_M^2} = \frac{E(r_i) - r_f}{2\,Cov(r_i, r_M)}$$

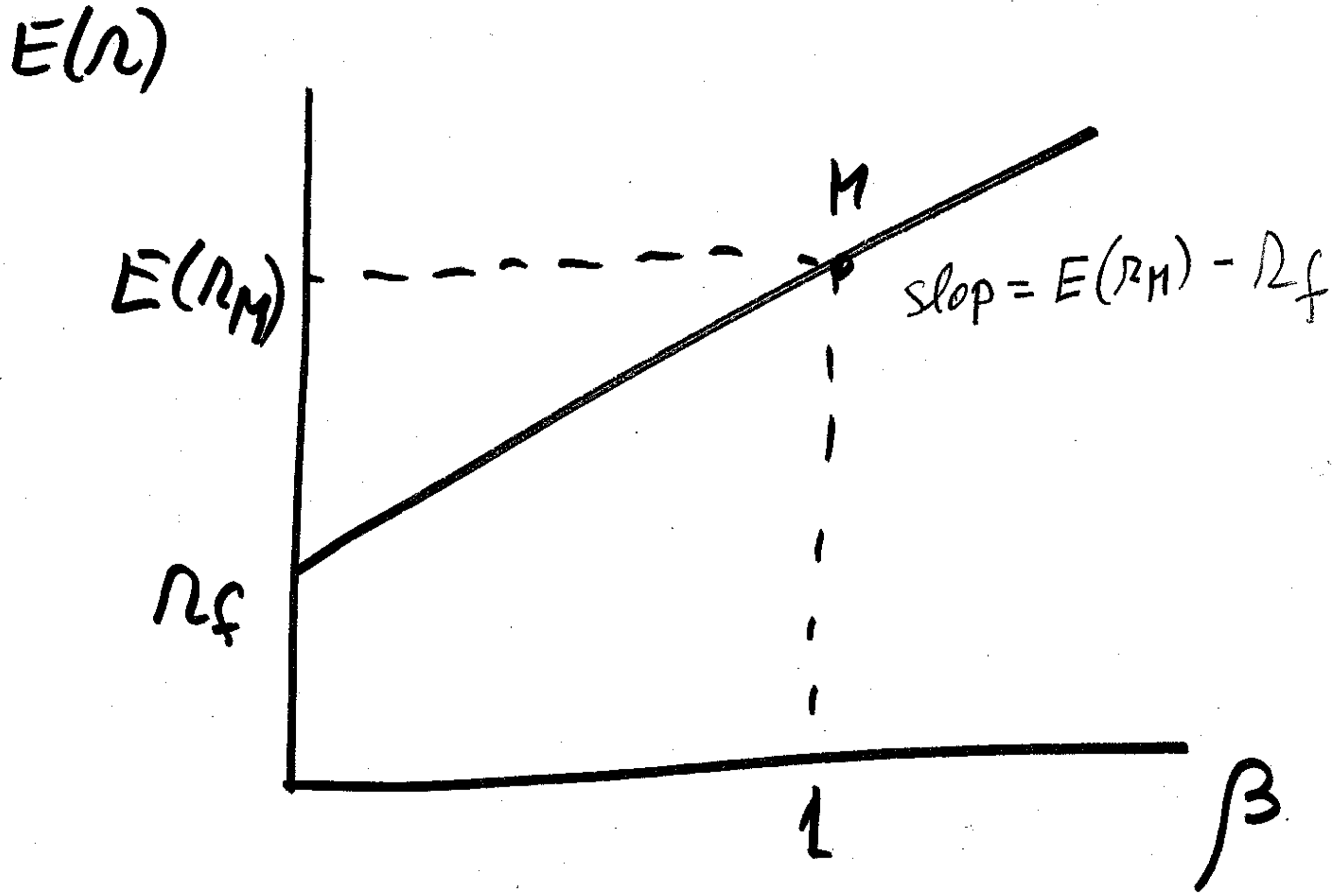
or

$$E(r_i) - r_f = \beta_i (E(r_M) - r_f]$$

$$\beta_i = \frac{Cov(r_i, r_M)}{\sigma_M^2}.$$

This must be true for all stocks in the market. We can trace this linear relation in β_i. It is called <u>security market line.</u>

Security Market Line



All stocks are lined up along this line which measures the cost of risk

$$\frac{E(r_M) - r_f}{\sigma_M} = \text{Market Price of Risk}$$

Expected return per unit of risk.

Security Market Line (contd.)

Slope of Security Market Line or the Market Cost of Risk is determined by the

- **Feasible set of securities**
- **The average risk aversion in the economy**

$$r_i - r_f = \beta_i(r_M - r_f) + \varepsilon_i$$

excess return on i	excess return on market

Computing the implied variances we find the basic equation which describes the total risk on holding an asset:

$$\sigma^2_{r_i} = \beta_i^2 \sigma_M^2 + \sigma^2_{\varepsilon_i} + 2\beta_i \mathrm{Cov}(r_M, \varepsilon_i)$$

Decomposition of Risk

The ε_i risk:

$E(\varepsilon_i) = 0$ $\quad\varepsilon_i$ is not systematic risk

$Cov(r_M, \varepsilon_i) = 0$ $\quad\varepsilon_i$ is security specific risk

Hence total risk of r_i Consist of

- $\beta_i^2 \sigma_M^2$ - variance due to Market Systematic Risks. This is the component of risk which cannot be eliminated by diversification since the market risk cannot be diversified.
- $\sigma_{\varepsilon_i}^2$ - variance due to security specific risk. Idiosyncratic Risks can be reduced by holding many securities like i and hence by portfolio diversification.

Covariance of Security Returns

What about $Cov(r_i, r_j)$?

From the formula we know that

$Cov(\varepsilon_i, \varepsilon_j) = 0$ - since ε_i are specific.

Hence

$$Cov(r_i, r_j) = \beta_i \beta_j \sigma_M^2$$

Correlation among returns is induced by joint correlation with the market which is the source of all systematic risks.

CAPM: Summary of Assumptions

1. Quadratic Utility

2. Income only from assets: traded securities

3. Individuals are price takers

4. Single period investment horizon

5. No taxes, no transaction cost

6. Costless information available to all investors

7. Homogenous expectations

8. Investors are rational and optimize.

CAPM: Summary of Conclusions

1. All investors hold the same portfolios of risky assets. They differ by the proportion of risky assets in their own portfolio.

2. The uniform portfolio is the market portfolio.

3. Risk premium on an individual security is proportioned to the market risk in security i

$$Cov(r_i, r_M)$$

4. A portfolio consisting of a single security is most unlikely to be optimal.

Examples of Two Applications

1. **What return to expect?**

Statistical analysis of the market reveals that

$E(r_M) = .15$, $\sigma(r_M) = .16$, $r_f = .05$.

You consider a portfolio r which has been shown to have

$$Cov(r, r_M) = .035.$$

What do you expect the return on the portfolio to be?

Solution

Since $\beta = .035/(.16)^2 = .035/.0256 = 1.367$ **the basic CAPM formula shows that**

$$E(r) = .05 + (1.367)(.15 - .05) = .186.$$

Hence you should expect a 18.6% return since the security is rather risky.

2. Which Portfolios are Efficient?

Assume the same market information as in 1. Now consider the following two portfolios

(1) Portfolio y with

$E(r_y) = .13$, $Var(r_y) = .02000$

(2) Portfolio v with

$E(r_v) = .09$, $Var(r_v) = .004096$.

Are these two portfolios efficient?

Solution: We know that the market price of risk, which applies to all efficient portfolios, is

$$\frac{E(r_M) - R_f}{\sigma_M} = \frac{.10}{.16} = .625.$$

Test if the two portfolios satisfy the formula

$$\frac{E(r_y) - r_f}{\sigma_y} = \frac{.13 - .05}{\sqrt{.02}} = \frac{.08}{.1414} = .5657 < .625 \text{ INEFFICIENT}$$

$$\frac{E(r_v) - r_f}{\sigma_v} = \frac{.09 - .05}{\sqrt{.004096}} = \frac{.040}{.064} = .625 \qquad \text{EFFICIENT}$$

6.5 Factor Models

The idea that market is the source of risk leads to search for the underlying causes of market risk. If we could study them directly we would have a better explanation of the structure of market risk.

- **Also , mean-variance approach requires substantial information**
- **with n securities we need**

  n - means

  n - variance

  $$\frac{n(n-1)}{2} \text{ Covariances.}$$

With n = 1,000 we need 501,500 parameters!!

- **Factor Models focus on the essential sources of uncertainty**
- **Reduce information requirement**
- **Simplify reasoning**
- **Are not based on GE considerations**

A Single Factor Model

n - Securities
r_i - Return on the i'th security.
We postulate the return is determined by

$$r_i = a_i + b_i f + e_i \quad , \qquad i = 1,2,\ldots,n.$$

$$E(e_i) = 0 \;, \quad E(e_i e_j) = 0 \quad , \; E[(f - \bar{f})e_i] = 0$$

$\sigma^2_{e_i}$ = **variance of e_i is assumed known**
a_i = intercepts,
b_i = factor loadings

f = the source of uncertainty is common to all securities i
e_i = an idiosyncratic component of security i

How Many Parameters?

$$\bar{r}_i = a_i + b_i \bar{f}$$

$$\bar{\sigma}_i^2 = b_i^2 \sigma_f^2 + \sigma_{e_i}^2$$

$$\sigma_{ij} = b_i b_j \sigma_f^2 \qquad i \neq j$$

$$b_i = \frac{\text{cov}(r_i \cdot f)}{\sigma_f^2}$$

We need 3n + 2 parameters a_i, b_i, $\sigma_{e_i}^2$, $\bar{f}$, σ_f^2 and this is much less than the number specified earlier.

Portfolio Parameter Simplification

Form portfolio $w = (w_1, w_2, \ldots, w_n)$ with return r

$$r = \sum_{i=1}^{n} r_i w_i \quad , \quad \sum_{i=1}^{n} w_i = 1.$$

Now, since

$$r_i = a_i + b_i f + e_i$$

we must have

$$r = \underbrace{\sum_{i=1}^{n} w_i a_i}_{= a} + \underbrace{\sum_{i=1}^{n} w_i b_i f}_{+ bf} + \underbrace{\sum_{i=1}^{n} w_i e_i}_{+ e}$$

Hence, the riskiness of any portfolio is a sum of two components:

bf = determined by the single source of risk
e = determined by diversification of many idiosyncratic effects which we now study.

Diversification when $E(e_i e_j) = 0$

$$\sigma_e^2 = E(e^2) = E\left(\sum_{i=1}^{n} w_i^2 e_i^2\right) = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$$

If - for simplicity - $\sigma_{e_i}^2 = s^2$ and $w_i = \frac{1}{n}$

the variance of the portfolio is

$$\sigma_e^2 = \frac{1}{n} s^2 \to 0 \quad \text{as} \quad n \to \infty.$$

In general the variance of r is σ and

$$\sigma^2 = b^2 \sigma_f^2 + \sigma_e^2$$

We observe, however, that

- the risk of f is not diversifiable
- the idiosyncratic risks of e_i are diversifiable since σ_e^2 can be reduced close to 0 if n is large.

Multifactor Models

In many applications we find several factors. These are macro-economic variables affecting security prices but which are not diversifiable. We write these in the form

$$r_i = a_i + b_{1_i} f_1 + b_{2_i} f_2 + e_i$$

(b_{1i}, b_{2i}) are factor loadings. We compute the moments in the same way as above, for example

$$\mathrm{cov}(r_i, f_1) = b_{1_i} \sigma_{f_1}^2 + b_{2_i} \sigma_{f_1 f_2}$$

$$\mathrm{cov}(r_i, f_2) = b_{1_i} \sigma_{f_1 f_2} + b_{2_i} \sigma_{f_2}^2$$

Other moments can also be computed.

CAPM as a Factor Model

The one factor model can be written as

$$* \qquad r_i - r_f = \alpha_i + \beta_i\,(r_M - r_f) + e_i$$

where for the general CAPM

$$\alpha_i = 0$$

hence

$$E(r_i) - r_f = \beta_i\,(E(r_M) - r_f).$$

This is familiar.

α_i = reflects a stock's deviation from the "correct" pricing. That is α_i is a measure of market "mispricing"

From (*) we deduce that

$$cov(r_i, r_M) = \beta_i \sigma_M^2$$

hence

$$\beta_i = \frac{Cov(r_i, r_M)}{var(r_M)}.$$

What additional restrictions can be imposed on the intercepts a_i and the factor loadings b_i of any factor model? This is our next topic.

Arbitrage Pricing Theory (APT)

Consider the simple factor model

$$r_i = a_i + b_i f \quad i = 1, 2...$$

without an error term.

Given there is only one factor, what restrictions can we place on the a_i and b_i? Consider a second security j

$$r_j = a_j + b_j f. \quad b_j \neq b_i.$$

Form a portfolio (w, 1 - w) with return

$$r = [wa_i + (1 - w)a_j] + [wb_i + (1 - w)b_j]f.$$

Now select w so that

$$wb_i + (1 - w)b_j = 0$$

$$\Rightarrow \quad w = \frac{b_j}{b_j - b_i} .$$

APT (Continue)

Then this portfolio is riskless hence

$$r = wa_i + (1 - w)a_j = \frac{a_i b_j}{b_j - b_i} - \frac{a_j b_i}{b_j - b_i}.$$

Hence all portfolios constructed as this was will have the same return λ_0

$$\lambda_0 = \frac{a_i b_j - a_j b_i}{b_j - b_i}$$

or

$$\frac{a_j - \lambda_0}{b_j} = \frac{a_i - \lambda_0}{b_i} \quad \textbf{for all } i,j$$

That is

$$\frac{a_i - \lambda_o}{b_i} = C = \text{constant}$$

APT (Continue)

Hence, for all (a_i,b_i) we must have the relationship

$$a_i = \lambda_0 + cb_i$$

If there is a riskless asset then

$$\lambda_0 = r_f = a_i \quad , \quad b_i = 0.$$

But then $\bar{r}_i = E(r_i) = a_i + b_i\bar{f}$

$$= \lambda_0 + cb_i + b_i\bar{f}$$

$$= \lambda_0 + b_i\lambda_1$$

where $\lambda_1 = c + \bar{f}$.

<u>Conclusion:</u> There exist two constants (λ_0, λ_1) which fix the expected return of the security. You cannot choose both a_i and b_i!!

<u>Example</u>: Suppose f is the S&P 500 if $a_i = 0.50$ then $b_i < 0$.

Theorem on Simple APT

Suppose there are n assets with risk determined by m < n factors,

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j.$$

Then, there are m constants

$$(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m)$$

such that for each i

$$\bar{r}_i = \lambda_0 + \sum_{j=1}^{m} b_{ij} \lambda_j.$$

The return on securities is then determined by the m market parameters and by factor loadings b_{ij}.

6E. CAPM and Corporate Finance

We discussed two issues:

(1) **The NPV criterion** for project evaluation

$$\mathbf{NPV} = \mathbf{C_0} + \sum_{t=1}^{T} \frac{\mathbf{E(C_t)}}{\mathbf{(1 + r)^t}}$$

(2) **CAPM Theory** about expected return on equity required by market to take risk.

Now we argue that:

A. (2) gives interest rate in (1) for pure equity firm.

B. For firm with equity and debt, project cost of capital is a weighted average of cost of debt and cost of equity.

<u>Hence CAMP theory is a foundation for corporate finance.</u>

E.1 Cost of Equity Capital

Under CAPM the expected rate of return on equity

$$\mathbf{E(R_j) = R_F + \beta_j \times [E(R_M) - R_F]}$$

United Semiconductor considers a new project to increase its size. New project is like the existing firm. We know

- stock has a $\beta = 1.3$
- it is 100% equity financed
- Risk free rate = 5%
- Market risk premium is 8%

$$\mathbf{r_S = 5\% + 1.3 \times (8) = 15.4\%}.$$

Hence $\mathbf{E(R_j) = 15.4\%}$, projects discounted at 15.4%

We can then either

- Evaluate projects at 15.4% and reject all with negative NPV

OR

- Accept any project with internal rate above 15.4%

Examples of Causes of Beta

- Business fluctuations.
- Some subtle. An example:

Consider two methods of industry production:

	Method A	Method B
Fixed Cost:	\$1,000 a year	\$2,000 a year
Variable cost:	\$8 per unit	\$6 per unit
Price of product:	\$10 per unit	\$10 per unit
Margin:	\$2 = \$10-\$8	\$4 = \$10-\$6

Fluctuations in sales imply that

In A: Change of100 units changes profits by \$200
In B: Change of100 units changes profits by \$400
(i) Profits of firms using Method B fluctuate more
(ii) In B profits respond more to business conditions

We know that using debt increases risk. We now want to see how it influences corporate decisions.

Return and Cost of Leveraged Projects

Assets A, B with cost rates or return rates r_A and r_B. What is the cost rate or return rate on A+B?

FACT:

$$\mathbf{r_{A+B} = \frac{r_A A}{A+B} + \frac{r_B B}{A+B}}$$

Take Expectations

$$\mathbf{E(r_{A+B}) = E(r_A)\frac{A}{A+B} + E(r_B)\frac{B}{A+B}}$$

$$\mathbf{Cov(r_{A+B}, r_M) = Cov(r_A, r_M)\frac{A}{A+B} + Cov(r_B, r_M)\frac{B}{A+B}}$$

Hence

$$\beta_{A+B} = \beta_A \frac{A}{A+B} + \beta_B \frac{B}{A+B}.$$

<u>The Effect Leverage</u>: Increased risk of project.

- Distinguish between

  Equity beta - risk to equity return

  Asset beta - risk to total asset return

We know

$$\beta_{Asset} = \beta_{Debt} \times \frac{\textbf{Debt}}{\textbf{Debt} + \textbf{Equity}} + \beta_{Equity} \times \frac{\textbf{Equity}}{\textbf{Debt} + \textbf{Equity}}$$

- Universally agree that beta of debt is very small
- We assume here beta of debt = 0
- Hence, using notation Debt≡B and Equity≡S

$$\beta_{Asset} = \beta_S \times \frac{S}{B+S} \quad \Rightarrow \quad \beta_S = \beta_{Asset} \times \left(1 + \frac{B}{S}\right)$$

Hence $\beta_S > \beta_{Asset}$: leverage increase risk!

Now we return to the issue of **<u>cost of capital.</u>**

E.2 Cost of Capital

Using both debt and equity, the cost of capital on a jointly financed project is simply (without tax)

$$r_{WACC} = r_S \times \frac{S}{B + S} + r_B \times \frac{B}{B + S}.$$

Since interest cost are deductible from corporate tax:

Actual cost of debt after tax $= r_B \times (1 - T_C)$

hence

$$r_{WACC} = r_S \times \frac{S}{B + S} + r_B \times (1 - T_C) \times \frac{B}{B + S}.$$

Example

United Telecom has $40 millions of debt and stock market equity value of $60 millions. (3 million shares at $20 a share)

- Cost of debt to firm is 6%
- beta of stock is 1.5
- corporate tax of 35%
- Market risk premium of 7%
- Treasury bill rate of 2%

We compute the following.

By CAPM

$$\mathbf{r_S = 2\% + 1.5 \times (7\%) = 13.5\%.}$$

Effective cost of debt

$$\mathbf{r_B(1 - T_C) = 6 \times (1 - 0.35) = 3.9\%.}$$

Aggregate cost of capital of the firm

$$\mathbf{r_{WACC} = 13.5 \times \frac{60}{100} + 3.9 \times \frac{40}{100} = 9.66\%.}$$

Project Evaluation

Any project which is financed jointly with debt and equity needs to earn at least 9.66% to be accepted. The NPV is then computed by

$$\mathbf{NPV = C_0 + \sum_{t=1}^{T} \frac{E(C_t)}{(1 + r_{AWCC})^t}.}$$

CAPM and Debt Valuation: A Problem

Ignoring taxes and assuming $\beta_B \neq 0$, CAPM offers a symmetric view of debt and equity. It proposes both

$$r_S = r_F + \beta_S \times [r_M - r_F]$$

$$r_B = r_F + \beta_B \times [r_M - r_F].$$

Since

$$r_{WACC} = r_S \times \frac{S}{B + S} + r_B \times \frac{B}{B + S}$$

$$\beta_{Asset} = \beta_{Debt} \times \frac{Debt}{Debt + Equity} + \beta_{Equity} \times \frac{Equity}{Debt + Equity}$$

we have

$$r_{WACC} = r_F + \beta_{Asset} \times [r_M - r_F].$$

Problem:

- β_B is small, at most 0.1 for all bonds.
- $r_F \approx 2\%$ and $r_M \approx 8\%$
- CAPM: all bonds should yield about 2.8%.
- Rejected by data, Yields vary from 2% riskless rate to 15% of low grade bonds. **Very Large Default Risk Premia**.

Bond Pricing (Cont)

- Bond risks are dominated by default risks
- Default involves Moral Hazard and hence difficult to insure against
- Markets cannot diversify all default risks
- The event of "default" itself is catastrophic: once defaulted, a firm is out of the market
- This does not fit well into the CAPM setup
- Hence, it is a problem for CAMP theory

To avoid these difficulties in our class

(i) we shall assume $\beta_B = 0$

(ii) We recognize that the equation

$$r_B = r_F + \beta_B \times [r_M - r_F]$$

has a limited value

(iii) In most applications accept cost of debt to be exogenous to the firm.

E.3 Modigliani Miller Theory: No Taxes

- Capital Structure: consists of equity and debt

⇒ Value of firm = Value of Equity +Value of Debt

Question: Can you change value of the firm by altering the capital structure?

Applications:

- Taking debt to pay equity holders
- Use leverage to restructure
- Use debt to break up firm
- Mergers and other combinations

E3.1 Maximization of Firm Value

- Debt obligation is a fixed payment of interest each period.
- Hence, for each debt level, the equality

  Value = Equity + Debt

Imply: to maximize firm value managers maximize value of Equity which is what stockholders want.

E3.2 Effect of Leverage on Return

United Fruit Inc.

- No debt;
- May issue debt to pay equity investors.
- Current and proposed capital structures:

	Current	Proposed
Assets	$8,000	$8,000
Debt	0	$4,000
Equity at Market Value	$8,000	$4,000
Interest rate	10.0%	10.0%
Market Value/ Share	$20	$20
Shares outstanding	400	200

How do earnings vary over time?
Assume risk is described by three possible states of

"Recession" , **"As Expected"** and **"Expansion."**

CASE 1: All Equity Firm

	Recession	Expected	Expansion
Return on Assets	5%	15%	25%
Earnings	$400	$1,200	$2,000
Equity Return	5%	15%	25%
Earning\share	$1.00	$3.00	$5.00

CASE 2: Proposed Leveraged Firm

	Recession	Expected	Expansion
Return on Assets	5%	15%	25%
Earnings	$400	$1,200	$2,000
Interest	-$400	-$400	-$400
Earning - interest	0	$800	$1,600
Equity Return	0	20%	40%
Earning\share	0	$4.00	$8.00

Assume: three states have the same probabilities

Hence: Expected Earning\share is

- $4 for levered firm
- $3 for all equity firm

It looks like leverage is better for stockholders since it increased their expected return!!

NO: Leverage also increased risk.
Standard deviation of earnings per share

If firm is all equity $= \sqrt{8}$

If firm is levered $= \sqrt{32}$.

Changed capital structure changes riskiness of firm's return to shareholders. Risk averse investors may prefer an all equity firm.

The Main Issue: Can change in capital structure change the value of combined firm consisting of Equity and Debt.

Why? Use money received for stock to buy bonds. By owning both stock and bond you have the same earning as before. Increased combined value means you get an added value with no change in earnings.

Modigliani and Miller Proposition I (no taxes)

The Value of the combined levered firm is the same as the value of the all-equity firm.

Modigliani Miller Prop. I (no taxes)

Consider two strategies by a shareholder:

Strategy 1: **If Company is All Equity**

- Borrow $2,000 from Bank at 10% interest
- Add your own investment of $2,000
- Buy 200 shares at $20\share = $4,000

Outcome on $2,000 Investment

	Recession	Expected	Expansion
Earnings per 200 sh. of All Equity Firm	1×200 =$200	3×200 =$600	5×200 =$1,000
Interest 10% of 2,000	-$200	-$200	-$ 200
Earning - interest	0	$400	$ 800

Strategy 2: **If Company is Levered**

- Use your own $2,000
- Buy 100 shares at $20\share = $2,000

Outcome on $2,000 Investment

	Recession	Expected	Expansion
Earnings p/s of All Equity Firm	0	$4	$8
Earning on 100 shares	0	$400	$800

Analysis of Outcomes

- Outcome for a shareholder is the same
- Two investment are the same
- The market will value these as the same
- If you like leverage you can do it yourself and do not need the firm to do it for you.
- An investor does not receive from the firm's leverage anything that he cannot do himself.
- Hence the firm cannot add value by leverage
- If you prefer all equity you undo the firm's leverage by using money received for stock to buy the new bonds. Your earnings remain the same as in the all equity firm.
- Can investors borrow at the same cost of firms? *Yes, using margin account at brokers.*

The Modigliani -Miller Theory Implies

- If levered firms are too expensive, investors will sell them, buy unlevered firms and, on their own, do the leverage by borrowing
- If all equity firms are too expensive, investors will sell them, buy levered firms and undo the firm's leverage effect by buying bonds.

Modiglainai -Miller II: Required Equity Return

Recall basic equation

$$r_{WACC} = r_S \times \frac{S}{B + S} + r_B \times \frac{B}{B + S}.$$

In the example below

- The cost of borrowing is 10%
- the expected return on equity r_S is

All-equity firm: (1/3)5%+(1/3)15%+(1/3)25% = 15%

Leveraged firm: (1/3) 0 +(1/3)20%+(1/3)40% = 20%

Hence we find the surprising result that

All equity $r_{WACC} = 15\% \times \frac{8,000}{8,000} + 10\% \times \frac{0}{8,000} = 15\%$

Leveraged $r_{WACC} = 20\% \times \frac{4,000}{8,000} + 10\% \times \frac{4,000}{8,000} = 15\%$

Modigliani-Miller Proposition I implies the cost of capital for the combined firm remains the same!!

Modiglainai -Miller II: equity return r_S (cont)

Now define

r_0 = capital cost of an all equity firm.

- In a world without taxes $r_0 = r_{WACC}$
- Hence

$$r_0 = r_S \times \frac{S}{B + S} + r_B \times \frac{B}{B + S}.$$

Multiply both sides by (B+S)/S you get

$$\frac{B + S}{S} \times r_0 = r_S + r_B \times \frac{B}{S}.$$

Rearranging demonstrates what r_S must be.

Modigliani-Miller Proposition II (no taxes)

$$\mathbf{r_S = r_0 + \frac{B}{S} \times (r_0 - r_B)}$$

The required equity return is a linear in (B/S).

An Example

United Semi Inc. expects to earn $10 millions in perpetuity

- Earnings are paid as dividends
- There are 10 million shares
- Hence dividend per share is $1
- $r_0 = 10\%$
- Hence: stock value is $100 million since cash flow is capitalized at 10%

United Semi developes a new chip which will raise net cash flow by $1 million, requiring a new plant which cost $4 million. This is announced publically.

Effect of new Chip on Firm's value:

$$\text{NPV of Plant} = -4 + \frac{\$1\,\text{million}}{0.1} = \$6\,\text{million}$$

- value of United's stock rises to $106 million
- with 10 million shares a share price rises to $10.60
- Efficient market theory says it is immediate.
- The new plant can be financed by stock or by debt with cost of debt at 6%.

We analyze now these two options.

Equity Financing

- Firm needs \$4 million
- At \$ 10.60 it sells 377,358 new shares
- Number of shares rises to 10,377,358 shares
- The cash is paid to the contractor who builds the plan instantly (to avoid additional discounting).

Final Asset Structure of United Semi Inc.

Old Assets PV of New Plant	\$100 million \$ 10 million	Equity	\$110 million (10,377,358 shares)
Total Value	\$110 million	Total Value	\$110 million

- Note: cost of \$4 million is replaced by the present value of new plant
- Efficient markets says stock price remains \$10.60
- Expected annual cash flow to stockholders is \$11 million
- Expected return to stockholders is now

$$r_S = \frac{\$11 \text{ million}}{\$110 \text{ million}} = 10\%$$

Conclusion: $r_S = r_0 = 10\%$ since firm is all-equity.

Debt Financing

- Stock price is at $10.60 after Chip announcement
- National Semi borrows $4 million at 6% debt cost
- Agrees to pay annual $240,000 in perpetuity
- The plant is built instantly (for simplicity)

Final Asset Structure of United Semi Inc.

Old Assets	$100 million	Equity	$106 million
PV of New Plant	$ 10 million		(10,000,000 shares)
		Debt	$ 4 million
Total Value	$110 million	Total Value	$110 million

- Expected annual cash flow to stockholders is now

$10 million (old assets) + $1 million - $240,000 = $10,760,000

- Expected return to stockholders is now

$$r_S = \frac{\$10,760,000}{\$106,000,000} = 10.151\%$$

Conclusion: $r_S = r_0 + \frac{B}{S} \times (r_0 - r_B) = 10.151\%$

since $r_S = 10 + \frac{4}{106} \times (10 - 6) = 10.151\%$ **since the firm is now leveraged.**

E.4 Modigliani-Miller With Taxes

Crucial effect of tax: interest cost are deductible.

- **A company has $1 million Earnings Before Taxes and Interest (EBIT)**
- **Corporate tax rate of 35%**
- **$4 million of debt**
- **Cost of debt = 10%**

Net Earnings Eith and Eithout Debt

	WITHOUT	WITH
EBIT	$1,000,000	$1,000,000
Interst $r_B B$	0	(400,000)
Earnings before taxes	$1,000,000	600,000
Taxes	(350,000)	(210,000)
Earnings after taxes	650,000	390,000

$EAT = (EBIT - r_B B)(1 - T_C)$

Cash Flow to Stockholder and Bondholders	$650,000	$790,000

Debt is a mechanism to reduce share of IRS in profits.

Definition: Tax Shield is the reduction in taxes paid due to the presence of debt.

Modigliani-Miller With Taxes (cont)

- By definition: Interest = $r_B B$
- Reduction in taxes was \$140,000 which is
  \$140,000 = \$400,000×(0.35)
- In general it is = $T_C \times r_B \times B$
- Capitalize it by the cost of capital since risk of the gain $T_C \times r_B \times B$ is the same as interest on the debt.
- Hence the capitalized value is

$$\frac{T_C(r_B B)}{r_B} = T_C B$$

Value of Levered Firm

- All-equity firm's annual cash flow $= \text{EBIT} \times (1 - T_C)$
- Value of all-equity firm: $V_U = \dfrac{\text{EBIT} \times (1 - T_C)}{r_0}$

Modigliani-Miller Proposition I (with taxes)

$$\mathbf{V_L = \frac{EBIT \times (1 - T_C)}{r_0} + \frac{T_C r_B B}{r_B} = V_U + T_C B}$$

Value of Levered Firm Equals Value of Unlevered Firm plus Capitalized Tax Shield.

Expected r_S With Taxes (cont)

- Expected cash flow from asset

$$= \text{EBIT}\times(1 - T_C) + T_C(r_B B)$$

$$= V_U r_0 + T_C(r_B B)$$

- Expected cash flow to stocks and bonds

$$= S r_S + B r_B$$

These two must be the same, hence

$$S r_S + B r_B = V_U r_0 + T_C(r_B B).$$

Divide both sides by S, subtract Br_B to obtain

$$r_S = \frac{V_U}{S}\times r_0 - (1 - T_C)\frac{B}{S}\times r_B.$$

Use the identities $V_L = V_U + T_C B = S + B$ to conclude

$$\frac{V_U}{S} = 1 + (1 - T_C)\frac{B}{S}$$

hence we finally have the key result:

Modigliani-Miller II (with taxes)

$$r_S = r_0 + \frac{B}{S} \times (1 - T_C) \times (r_0 - r_B).$$

Once we derive cost of equity with taxes we can derive the weighted cost of capital of the firm.

$$r_{WACC} = r_S \times \frac{S}{V_L} + r_B \times (1 - T_C) \frac{B}{V_L}$$

where cost of debt is multiplied by $(1-T_C)$ to account for interest deductibility from taxation.

An Example (All numbers below are in millions)

- United Software Inc. is an equity-only company
- It expects to have 153.85 in EBIT in perpetuity
- $T_C = 0.35$ hence after tax Earnings = 100
- Value of United is 500 and there are 100 shares
- All earnings after tax and interest are paid out
- United is considering an issue of 200 in debt to purchase stock.
- $r_B = 10\%$.

Questions

Compute the following values after the bond issue:

V_L

r_S

r_{WACC}

United stock price..

Answers

$$r_0 = \frac{100}{500} = 0.20$$

hence

1. $$V_L = \frac{153.85 \times 0.65}{0.20} + 0.35 \times 200 = 500 + 70 = 570.$$

2. $$r_S = 20\% + \frac{200}{370} \times (1 - 0.35) \times (20 - 10) = 23.51\%$$

3. $$\mathbf{r_{WACC} = 23.51 \times \frac{370}{570} + 10 \times (0.65) \frac{200}{570} = 17.54}$$

4. Physical assets of United are worth 500. With 100 shares, stock price before the financing is $5

5. After the announced financing the value rises to 570 creating a 70 capitalized tax shield. It is an important concept to grasp.

6. At $5.70 United retires 200/(5.7)= 35.09 shares, leaving 64.91 outstanding. Their equity value is 370 = 570- 200 and price 370/(64.91) = 5.70 hence the stock price remains 5.70 after financing.

7. Stockholders recovered all the value added. It came at the start, before any purchase of stock.

8. Why not buy more stock and create more tax shield? Due to the rising risk of default which arises from fixed interest obligations. We do not cover the problem of default in this course.