5. Equilibrium Determination of Asset Prices and Interest Rate: Uncertainty

Equivalence (basic asset pricing)

I. Portfolio Equivalence

$$\alpha_1 = d_1(1)\theta_1 + d_2(1)\theta_2$$
$$\alpha_2 = d_1(2)\theta_1 + d_2(2)\theta_2$$

In matrix notation

$$\alpha = A\theta$$

II. Pricing Equivalence

$$q_1 = d_1(1)p_1 + d_1(2)p_2$$
$$q_2 = d_2(1)p_1 + d_2(2)p_2$$

In matrix notation

$$q^T = p^T A.$$

When Does Equivalence Hold?

- When number of securities equals number of states, **and**
- Payoff matrix can be inverted.

In that case market is **COMPLETE**

- In our economy $A = \begin{pmatrix} 1 , & 1 \\ 1 , & 2 \end{pmatrix} \equiv (d_1 , d_2)$

and these conditions are met.

General Equilibrium

- Define consumption demand or portfolio demand in terms of state prices
- Equilibrium holds when in all states s

$$\textbf{Supply in s} = \sum_{h=1}^{H} e_s^h = \textbf{Demand in s} = \sum_{h=1}^{H} c_s^h$$

State prices are the solutions of these equations.

Once we have state prices we infer security prices.

Interpreting State Prices

- State price tells you the cost of consumption in state s in terms of state 0 consumption
- A state price for state s is also the price of a composite portfolio which pays 1 unit of consumption in state s and nothing in all other states
- Securities are valued in terms of state prices

A Portfolios which pays like AS

- Consider the case of a bond $d_B = (1,1)$ and a stock $d_S = (2, 1)$ (different from previous example). What portfolio pays

(a) 1 unit in state 1 and nothing in state 2?

(b) 0 unit in state 1 and 1 unit in state 2?

Answer

(a) $1d_S - 1d_B = (1,0)$ cost: $p_1 = q_S - q_B$

(b) $2d_B - 1d_S = (0,1)$ cost: $p_2 = 2q_B - q_S$

Computing General Equilibrium

We return to the economy with two states and log utility. Now it has two agents and we need to keep track of their identity.

We assume again no impatience so $\beta = 1$. Also

$$U^h(c_0, c_1, c_2) = \log(c_0) + \pi_1 \log(c_1) + \pi_2 \log(c_2)$$

$$\pi_1 = 0.1 \;,\; \pi_2 = 0.9.$$

Example 1:No Aggregate Uncertainty

The first person has incomes (endowments) of

$$e_0^1 = 100,\ e_1^1 = 0,\ e_2^1 = 100\,.$$

The second person has initial incomes

$$e_0^2 = 50,\ e_1^2 = 150,\ e_2^2 = 50\,.$$

Total economy's income in state 0 = 150

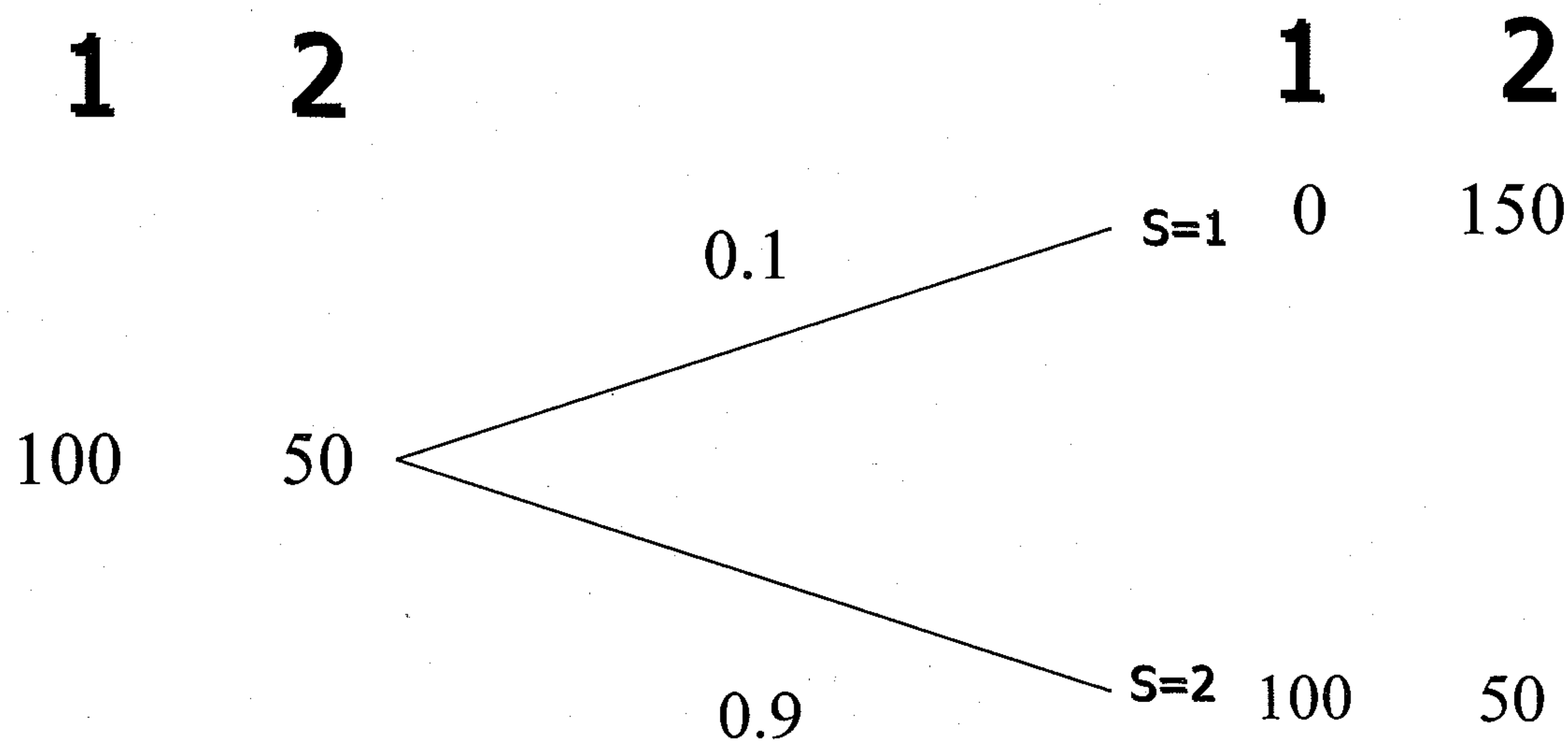
Total economy's income in state 1 = 150

Total economy's income in state 2 = 150.

Hence there is no risk in the market as a whole.

Individuals face risk but not the entire economy.

Individual endowments



Example 1

- Utility functions are

$$U^1(c_0, c_1, c_2) = \log(c_0) + 0.1 \cdot \log(c_1) + 0.9 \cdot \log(c_2)$$
$$U^2(c_0, c_1, c_2) = \log(c_0) + 0.1 \cdot \log(c_1) + 0.9 \cdot \log(c_2)$$

Definition of equilibrium I

- Given state prices, define individual demand as $c(p_1,p_2)$

$$c^h(p_1,p_2) = \arg\max U^h(c_0,c_1,c_2) \text{ s.t } c_0 - e_0^h + p_1(c_1 - e_1^h) + p_2(c_2 - e_2^h) = 0$$

EQUILIBRIUM :

$$c_1^1(p_1,p_2) + c_1^2(p_1,p_2) = e_1^1 + e_1^2$$

$$c_2^1(p_1,p_2) + c_2^2(p_1,p_2) = e_2^1 + e_2^2$$

Solving for equilibrium

- First solve for optimal consumption as a function of the state prices

$$c_1^1 = \pi_1 \frac{e_0^1 + p_1 e_1^1 + p_2 e_2^1}{2p_1} = 0.1\frac{100+100p_2}{2p_1}$$

$$c_1^2 = \pi_1 \frac{e_0^2 + p_1 e_1^2 + p_2 e_2^2}{2p_1} = 0.1\frac{50+150p_1+50p_2}{2p_1}$$

$$c_2^1 = \pi_2 \frac{e_0^1 + p_1 e_1^1 + p_2 e_2^1}{2p_2} = 0.9\frac{100+100p_2}{2p_2}$$

$$c_2^2 = \pi_2 \frac{e_0^2 + p_1 e_1^2 + p_2 e_2^2}{2p_2} = 0.9\frac{50+150p_1+50p_2}{2p_2}$$

Solving for equilibrium

- Now impose market clearing

$$c_1^1(p_1, p_2) + c_1^2(p_1, p_2) = e_1^1 + e_1^2$$

$$c_2^1(p_1, p_2) + c_2^2(p_1, p_2) = e_2^1 + e_2^2$$

$$0.1\frac{100+100p_2}{2p_1} + 0.1\frac{50+150p_1+50p_2}{2p_1} = 150$$

$$0.9\frac{100+100p_2}{2p_2} + 0.9\frac{50+150p_1+50p_2}{2p_2} = 150$$

Solving for equilibrium

$$0.1\frac{100+100p_2}{2p_1}+0.1\frac{50+150p_1+50p_2}{2p_1}=150$$

$$0.9\frac{100+100p_2}{2p_2}+0.9\frac{50+150p_1+50p_2}{2p_2}=150$$

$$\Rightarrow$$

$$150\frac{2p_1}{0.1}=100+100p_2+50+150p_1+50p_2=150+150p_1+150p_2$$

$$150\frac{2p_2}{0.9}=100+100p_2+50+150p_1+50p_2=150+150p_1+150p_2$$

$$\Rightarrow p_1=0.1 \qquad p_2=0.9$$

Why do state prices equal probabilities?

$$\pi_1 \frac{e_0^1 + p_1 e_1^1 + p_2 e_2^1}{2p_1} + \pi_1 \frac{e_0^2 + p_1 e_1^2 + p_2 e_2^2}{2p_1} = e_1^1 + e_1^2$$

$$\pi_2 \frac{e_0^1 + p_1 e_1^1 + p_2 e_2^1}{2p_2} + \pi_2 \frac{e_0^2 + p_1 e_1^2 + p_2 e_2^2}{2p_2} = e_2^1 + e_2^2$$

$$e_0^1 + e_0^2 + p_1(e_1^1 + e_1^2) + p_2(e_2^1 + e_2^2) = \frac{2p_1}{\pi_1}(e_1^1 + e_1^2)$$

$$e_0^1 + e_0^2 + p_1(e_1^1 + e_1^2) + p_2(e_2^2 + e_2^2) = \frac{2p_2}{\pi_2}(e_2^1 + e_2^2)$$

Stocks and bonds

- Now suppose that the two guys have to trade in stocks and bonds
- Equilibrium prices for the stock and the bond ensure that stock- and bond markets clear
- Continuing the example

$$d_1 = (1,1) \qquad d_2 = (1,2)$$

Stocks and bonds

- Given the equilibrium state prices p(1) and p(2) which we calculated, the equilibrium asset prices have to satisfy the fundamental theorem of asset pricing:

$$q_1 = d_1(1)\,p(1) + d_1(2)\,p(2) = 0.1 + 0.9 = 1$$

$$q_2 = d_2(1)\,p(1) + d_2(2)\,p(2) = 0.1 + 2 \cdot 0.9 = 1.9$$

Prices of stocks and bonds

$$q_1 = 0.1 + 0.9 = 1$$

$$q_2 = 0.1 + 2 \cdot 0.9 = 1.9$$

- Saying that q_1 and q_2 are equilibrium asset prices means: If in this economy both agents see these prices and choose their utility maximizing portfolios markets for stocks and bonds must clear
  - Whatever one guy sells, the other has to buy, I.e. sum of portfolios is zero !

Equilibrium in the asset markets

- Clearly, if stock price and bond price follow from the state prices, the optimal consumption given stock and bond price has to be the optimal consumption given these state prices

$$c_1^1 = 0.1\frac{100+100\cdot 0.9}{2\cdot 0.1} = 95$$

$$c_2^1 = 0.9\frac{100+100\cdot 0.9}{2\cdot 0.9} = 95$$

$$c_1^2 = c_2^2 = 0.1\frac{50+0.1\cdot 150+0.9\cdot 50}{2\cdot 0.1} = 55$$

Optimal portfolio: Agent I

- To achieve his optimal consumption the first guy has to buy a portfolio which pays him 95 dollars in the first state and -5 dollars in the second.

$$\theta_1^1 + \theta_2^1 = 95$$

$$\theta_1^1 + 2\theta_2^1 = -5$$

$$\Rightarrow \theta_2^1 = -100 \text{ and} \theta_1^1 = 195 .$$

Optimal portfolio: Agent II

- To achieve his optimal consumption the second guy has to buy a portfolio which pays him -95 dollars in the first state and 5 dollars in the second state.

$$\theta_1^2 + \theta_2^2 = -95$$

$$\theta_1^2 + 2\theta_2^2 = 5$$

$$\Rightarrow \theta_2^2 = 100 \text{ and } \theta_1^2 = -195.$$

Asset markets clear

- If we find state prices which ensure that in all second period states aggregate consumption is equal to aggregate endowments, then the implied asset prices (which we get from the fundamental theorem) will ensure that the asset markets clear.
- Given the state prices, we immediately have the asset prices.

Consumption is constant across states

From the first order conditions we know

$$c_1 = \frac{\pi_1}{p_1} c_0 \quad \text{and} \quad c_2 = \frac{\pi_2}{p_2} c_0 \ .$$

Since state prices equal probabilities, it follows that consumption of each household is constant across all states. The agent is fully insured.

Diversifiable Risks

- Prices of both securities equal the expected value of their payoffs
- This is remarkable as agents are risk averse but *buy securities whose prices equal the expected value of their payoff*
- Hence, in equilibrium agents diversify away the risk of any individual security. What matters is how it contributes to the riskiness of their final consumption

Diversifiable Risks (cont.)

- Since there is no aggregate uncertainty agents can diversify their own individual risks regardless of individual security risks.
- **To put it differently**: owning a risky asset does not mean it increases the risk of consumption.

Returns

- Recall that the return to an investment is the payoff per dollar invested
- The return on a bond is the interest rate
- Returns on risky investments depend on the state of the world

$$1 + r_j(s) = \frac{d_j(s)}{q_j}$$

No aggregate uncertainty and reality

- When state prices equal the probabilities, expected returns of assets have to equal the interest rate

$$E(1+r_a) = \frac{E(d_a)}{q_a} = 1 \Rightarrow r_a = 0$$

- On average over the last 50 years the return on stocks was 7 percent higher than the return on bonds…

Aggregate uncertainty

- At different states of the world, aggregate output is different
- There are exogenous, random shocks to technology and to productivity.
- We want to examine how these aggregate shocks influence state prices and asset prices

No aggregate uncertainty: Example 2

Person 1 has incomes (or 'initial endowments') of

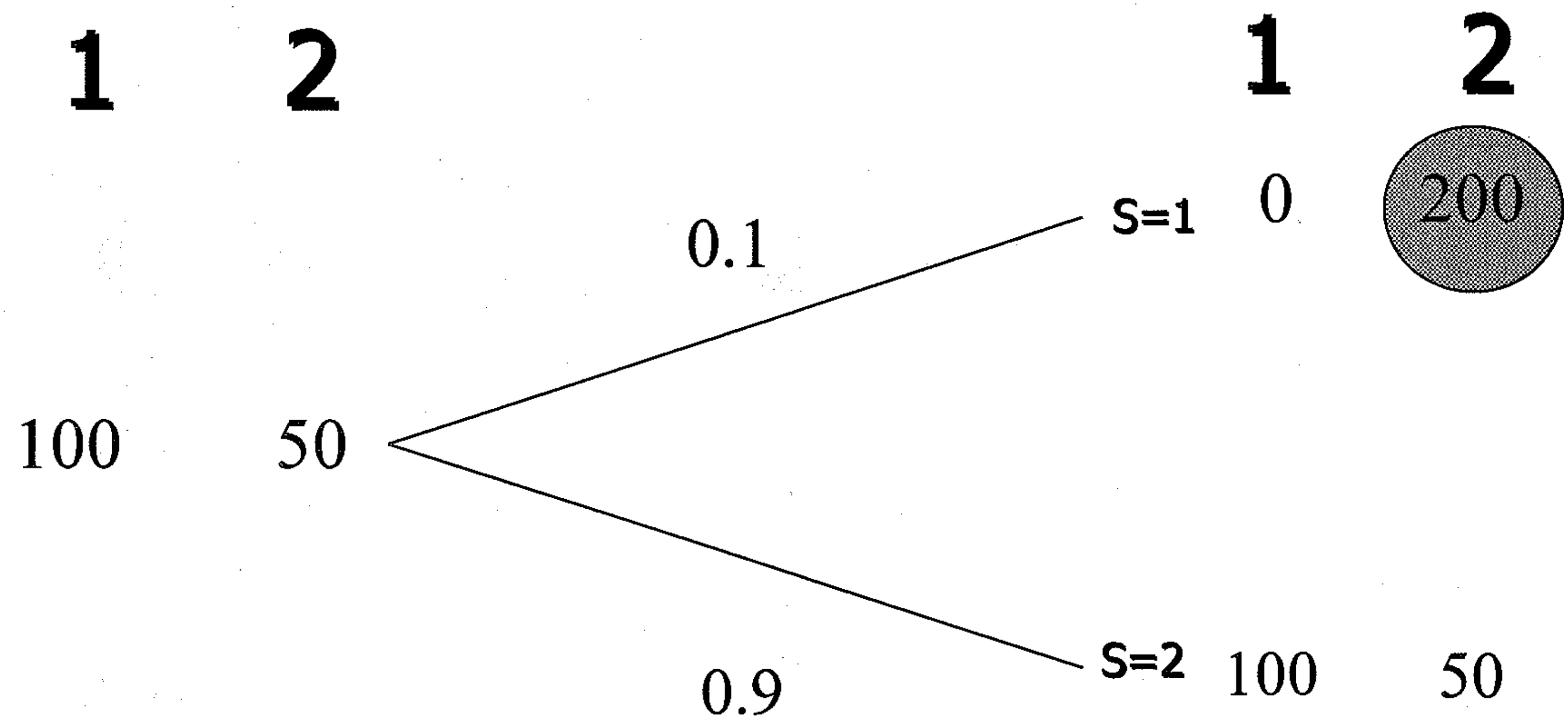
$$e_0^1 = 100,\ e_1^1 = 0,\ e_2^1 = 100.$$

Person 2 has initial endowments

$$e_0^2 = 50,\ e_1^2 = 200,\ e_2^2 = 50.$$

Total economy's income in state 0 = 150
Total economy's income in state 1 = 200
Total economy's income in state 2 = 150.
There is aggregate risk in the "Market" which cannot be diversified away.

Individual endowments



Solving for equilibrium

- First solve for optimal consumption as a function of the state prices

$$c_1^1 = \pi_1 \frac{e_0^1 + p_1 e_1^1 + p_2 e_2^1}{2p_1} = 0.1 \frac{100 + 100 p_2}{2p_1}$$

$$c_1^2 = \pi_1 \frac{e_0^2 + p_1 e_1^2 + p_2 e_2^2}{2p_1} = 0.1 \frac{50 + 200 p_1 + 50 p_2}{2p_1}$$

$$c_2^1 = \pi_2 \frac{e_0^1 + p_1 e_1^1 + p_2 e_2^1}{2p_2} = 0.9 \frac{100 + 100 p_2}{2p_2}$$

$$c_2^2 = \pi_2 \frac{e_0^2 + p_1 e_1^2 + p_2 e_2^2}{2p_2} = 0.9 \frac{50 + 200 p_1 + 50 p_2}{2p_2}$$

Solving for equilibrium

- Now impose market clearing

$$c_1^1(p_1, p_2) + c_1^2(p_1, p_2) = e_1^1 + e_1^2$$

$$c_2^1(p_1, p_2) + c_2^2(p_1, p_2) = e_2^1 + e_2^2$$

$$0.1\frac{100+100p_2}{2p_1} + 0.1\frac{50+200p_1+50p_2}{2p_1} = 200$$

$$0.9\frac{100+100p_2}{2p_2} + 0.9\frac{100+200p_1+50p_2}{2p_2} = 150$$

Solving for equilibrium

$$0.1\frac{100+100p_2}{2p_1}+0.1\frac{50+200p_1+50p_2}{2p_1}=200$$

$$0.9\frac{100+100p_2}{2p_2}+0.9\frac{100+200p_1+50p_2}{2p_2}=150$$

$$\Rightarrow$$

$$200\frac{2p_1}{0.1}=100+100p_2+50+200p_1+50p_2=150+200p_1+150p_2$$

$$150\frac{2p_2}{0.9}=100+100p_2+50+200p_1+50p_2=150+200p_1+150p_2$$

$$4000p_1=150+200p_1+150p_2$$

$$\frac{3000}{9}p_2=150+200p_1+150p_2$$

Solving for equilibrium

$$4000 p_1 = 150 + 200 p_1 + 150 p_2$$

$$\frac{3000}{9} p_2 = 150 + 200 p_1 + 150 p_2$$

$$\Rightarrow$$

$$p_1 = \frac{150}{3800} + \frac{150}{3800} p_2$$

$$p_2 = \frac{9}{1650}(150 + 200 p_1)$$

$$p_2 = 0.9$$

$$p_1 = 0.075$$

Solving for equilibrium – asset prices

$$p_2 = 0.9$$

$$p_1 = 0.075$$

$$\Rightarrow$$

$$q_1 = 0.9 + 0.075 = 0.975$$

$$q_2 = 0.075 + 2 \cdot 0.9 = 1.875$$

Solving for equilibrium returns

$$q_1 = 0.9 + 0.075 = 0.975$$
$$q_2 = 0.075 + 2 \cdot 0.9 = 1.875$$

$$1 + r_f = \frac{1}{0.975} \Rightarrow r_f = 2.56\%$$

$$E(1 + r_2) = \frac{0.1 + 0.9 \cdot 2}{1.875} \Rightarrow E(r_2) = 1.33\%$$

Explaining State Prices

- For state 1 the state price is lower than the probability $p(1) = 0.075 < 0.1$
  - There are more commodities to consume in state 1 since the higher endowment in the state means higher productivity;
  - when supply goes up, the price declines;

- The second state price equals the probability but this would not be the case if there was impatience.

Explaining Asset Returns

- The interest rate is above zero **although there is no impatience**
- It is higher than zero since there may be growth in endowment hence productivity.
- Return on the stock is below the interest rate:

The stock pays more in state 2 than in state 1 but in state 2 the economy is poorer. The stock return is low because people pay a high price for its desirability. It compensates the owner for the risk by making high payment just when needed: when the economy is in the poor state.

Explaining asset returns

- The price of the stock seems too high compared to the price of the bond – after all the stock is risky
- The variance of dividends is not important. What matters is the covariance of dividends with aggregate output (aggregate endowments)

A Final Example

- Two states of the world, S = 2;
- Two firms traded on the stock market and a riskless bond:

Table Maker - $\mathbf{d_T = (50, 75)}$, Car Maker - $\mathbf{d_C = (150, 365)}$

Bond with payoff $\mathbf{d_B = (1,1)}$

- Agent 1 owns Table maker, Agent 2 owns Car Maker;
- Market income

  in state 1 $e_1^1 + e_1^2 = 200$

  in state 2 $e_2^1 + e_2^2 = 440$ with significant Market risk;
- The state probabilities are $\boldsymbol{\pi_1} =$ **0.40** and $\boldsymbol{\pi_2} =$ **0.60.**

Final Example (cont.)

- Two agents - households. Household 1 owns the Table company and has 200 units today, $e_0^1 = 200$. His utility is

$$U(c_0, c_1, \ldots, c_S) = c_0 + 0.8 \sum_{s=1}^{S} \pi_s \left(c_s - \frac{1}{1000} c_s^2\right).$$

Household 2 owns car company and has 200 units today. His utility is

$$U(c_0, c_1, \ldots, c_S) = c_0 + 0.8 \sum_{s=1}^{S} \pi_s \left(c_s - \frac{1}{1200} c_s^2\right)$$

The difference between these two agents is the parameter of the quadratic term. **They have different risk aversion**.

Demand as a Function of State Prices

Given state prices p_1, p_2, households solve

$$\max_{(c_0, c_1, c_2)} U^h(c_0, c_1, c_2)$$

such that

$$c_0 + p_1 c_1 + p_c c_2 = e_0 + p_1 e_1 + p_2 e_2 .$$

The first-order conditions are

$$1 - \lambda = 0$$

For agent 1: $0.8\pi_s(1 - \frac{c_s}{500}) - \lambda p_s = 0$ for $s = 1, 2$

For agent 2: $0.8\pi_s(1 - \frac{c_s}{600}) - \lambda p_s = 0$ for $s = 1, 2$.

Demand Functions (cont.)

Easy to solve for the demand functions since $\lambda = 1$:

For agent 1: $c_s^1 = 500 - 625\frac{p_s}{\pi_s}$ for $s = 1, 2$

For agent 2: $c_s^2 = 600 - 750\frac{p_s}{\pi_s}$ for $s = 1, 2$.

Equilibrium: sum of demands equals sum of endowments

$$\mathbf{c_s^1 + c_s^2 = e_s^1 + e_s^2}.$$

Therefore

$$500 - 625\frac{p_s}{\pi_s} + 600 - 750\frac{p_s}{\pi_s} = e_s^1 + e_s^2$$

Solve for Equilibrium prices

Since

$$\frac{p_s}{\pi_s} = \frac{1100 - e_s^1 - e_s^2}{1375}.$$

Solution for state prices is simple (when probabilities specified)

$$p_1 = \frac{.40(1{,}100 - 200)}{1375} = 0.2618$$

$$p_2 = \frac{.60(1{,}100 - 440)}{1375} = 0.2880$$

Solve prices of bond, Table and Car Stocks

$$q_B = 1(0.2618) + 1(0.2880) = 0.5498$$

$$q_T = 50(0.2618) + 75(0.2880) = 34.69$$

$$q_C = 150(0.2618) + 365(0.2880) = 144.39\,.$$

More interesting: Expected Returns of Securities

General definition of expected return: the ratio

$$\mathbf{E(1 + r_j) = \frac{E(d_j)}{q_j}}\,.$$

We compute these now.

Computing Security Expected Returns

$$r_B = \frac{1}{0.5498} - 1 = 81.8843\%$$

$$r_T = \frac{E(d_T)}{q_T} = \frac{65}{34.69} - 1 = 87.3739\%$$

$$r_C = \frac{E(d_C)}{q_C} = \frac{279}{144.39} - 1 = 93.3267\%$$

- Price of Car company is higher than Table company- no surprise - it pays much higher dividends!
- It may be a surprise that Car company's shares also carry a higher *expected rate of return* than the expected rate of return of the Table company. **Why is this the case?**

The Role of Covariance with Market Return in Determining Asset Return

- Answer: covariance of Car Maker returns with aggregate market income is large hence Car Maker shares are riskier.
- Risk is what risk averse agents do not like. Hence they want compensation for the extra risk of owning these shares.
- We now show that the rate of return on each security is determined by its covariance with the market portfolio.

- Define the random aggregate income in the second period as the "Market" portfolio

$$m(s) = e^1(s) + e^2(s).$$

- The price of the market portfolio is simply

$$\mathbf{q_m = q_T + q_C = 34.69 + 144.39 = 179.08}.$$

Covariance with Market Return (cont.)

The net return on the market portfolio is

$$E(r_m) = \frac{E(m)}{q_m} - 1 = \frac{.40(200) + .60(440)}{179.08} - 1 = 92.0929\%$$

Excess return of the market portfolio is

$$E(r_m) - r_f = 92.0929 - 81.8843 = 10.2086\%.$$

Risk of the Market Portfolio is

$$Var(r_m) = \frac{.40(200-344)^2 + .60(440-344)^2}{(179.08)^2} = .43106 \Rightarrow 43.106\%$$

Differential Rates of Return on Assets

- Object: explain difference in expected return across the securities in the market.
- Thus compute ratio of excess return of each security in proportion to the excess return of the market portfolio:

$$\frac{\mathbf{E(r_j) - r_f}}{\mathbf{E(r_m) - r_f}}.$$

The two ratios are

$$\frac{E(r_T) - r_f}{E(r_m) - r_f} = \frac{87.3739 - 81.8843}{92.0929 - 81.8843} = \frac{5.4896}{10.2086} = 0.53774$$

$$\frac{E(r_C) - r_f}{E(r_m) - r_f} = \frac{93.2267 - 81.8843}{92.0929 - 81.8843} = \frac{11.3424}{10.2086} = 1.11106\,.$$

Explaining Differential Rates of Returns

- Consider the ratio of covariance of a security return with market return and riskiness of the market portfolio:
-

$$\frac{\mathbf{Cov}(\mathbf{r}_j, \mathbf{r}_m)}{\mathbf{var}(\mathbf{r}_m)} \qquad \mathbf{j = T, C}$$

where

$$r_T + 1 = \frac{50}{34.69} \text{ with probability } 0.40, \ \frac{75}{34.69} \text{ with probability } 0.60$$

$$r_C + 1 = \frac{150}{144.39} \text{ with probability } 0.40, \ \frac{365}{144.39} \text{ with probability } 0.60$$

$$r_m + 1 = \frac{200}{179.08} \text{ with probability } 0.40, \ \frac{440}{179.08} \text{ with probability } 0.60$$

Basic CAPM explanation: Covariance with the market

- Carry out these computations to find:

$$\frac{\mathrm{Cov}(r_T, r_m)}{\mathrm{var}(r_m)} = 0.53774$$

$$\frac{\mathrm{Cov}(r_C, r_m)}{\mathrm{var}(r_m)} = 1.11106$$

- Exactly the same as the ratios of the excess returns.
- To compute a general formula define beta_j (**Not Impatience**)

$$\frac{\mathrm{Cov}(r_j, r_m)}{\mathrm{var}(r_m)} = \mathrm{beta}_j$$

CAPM Result (cont.)

We have discovered the general principle: for any security

$$\frac{E(r_j) - r_f}{E(r_m) - r_f} = \frac{Cov(r_j, r_m)}{var(r_m)} = \beta_j$$

Or

$$[E(r_j) - r_f] = \frac{Cov(r_j, r_m)}{var(r_m)}[E(r_m) - r_f].$$

This is the fundamental result of **Capital Asset Pricing theory**.
Our next task is to explore it in further detail.